Problem set # 8

Only turn in the questions or problems marked with a (♣) for grading. It is however recommended that you try as many questions as you can.

1. **Brownian local maxima (♣).**

   Let $0 \leq a < b$ be two real numbers and $B = (B_t, t \geq 0)$ a Brownian motion. Show that, almost surely, a is the limit of local maxima of B on (a, b). [Hint: Recall from the previous homework that B is not monotone on any interval.]

   The set of local maximal of B is then almost surely dense in $\mathbb{R}_{\geq 0}$. Show that this set is also almost surely countable.

2. **Poisson Kernel and Brownian motion (♣).**

   Let $B_t = (B_t^{(1)}, B_t^{(2)})$ be a two dimensional Brownian motion started at the point $(0, 1)$ in the plane.

   Let T be the first time that the random point B_t in the plane exits the upper half plane $H := \{(x, y) \in \mathbb{R}^2 \colon y > 0\}$. Check that T is almost surely finite and compute the distribution of the exit point $B_T = (B_T^{(1)}, 0)$ on the real line. What is this distribution called?

3. **Distribution of the first return to 0 after 1 (♣).**

   Let $B = (B_t, t \geq 0)$ be a Brownian motion and set $T = \inf\{t \geq 1 \colon B_t = 0\}$. Find the density of T.