Problem set # 7

Only turn in the questions or problems marked with a (♣) for grading. It is however recommended that you try as many questions as you can.

1. **Some basic finite dimensional computations (♣).** Let $(B_t, t \geq 0)$ be a standard Brownian motion.

   (a) Let $0 \leq s < t$. Compute $\mathbb{P}(B_s > 0, B_t > 0)$. [Use the independence of increments.]

   (b) For $t \geq 0$, write $X_t = \int_0^t B_s ds$ as a limit of Riemann sums and compute its distribution.

   (c) Compute $\mathbb{E}[B_1^2 B_2 B_3]$.

2. **Brownian bridge (♣).** In this problem we wish to construct the so-called *Brownian bridge*. Let $a < b \in \mathbb{R}$, $T > 0$ and let $(B_t, t \geq 0)$ a standard Brownian motion. Define

$$X_t := \frac{bt + a(T-t)}{T} + B_t - \frac{t}{T} B_T$$

   (a) Show that $(X_t, 0 \leq t \leq T)$ is a Gaussian process with continous paths and compute $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t X_s]$ for $s, t \in [0, T]$.

   (b) Prove that $X = (X_t, t \in [0, T])$ is independent of B_T.

   (c) Find continuous functions $f, g : [0, T] \to \mathbb{R}$ such that $X = (X_t, t \in [0, T])$ has the same distribution as

$$\left(\frac{bt + a(T-t)}{T} + f(t) B_{g(t)}, \quad t \in [0, T] \right).$$

3. **Brownian motion is nowhere monotone (♣).** In this problem we wish to show that, almost surely, a standard Brownian motion is not monotone on any non-trivial interval $[a, b]$ in $\mathbb{R}$.

   (a) Let $a < b$ in $\mathbb{Q}_{>0}$. Using the independence of increments and their distribution, show that

$$\mathbb{P}(B \text{ is non-decreasing on } [a, b]) = 0.$$

   (b) Using the density of $\mathbb{Q}$ in $\mathbb{R}$, conclude that B is almost surely non-monotone on any non-trivial interval in $\mathbb{R}$.

4. **Brownian motion quadratic variation (♣).** Let $(B_t, t \geq 0)$ be a standard Brownian motion and fix $t > 0$. For $n \geq 1$ let

$$\Delta_{n,k} = B_{t(k+1)2^{-n}} - B_{tk2^{-n}} \quad \text{for } 0 \leq k \leq 2^n - 1.$$

   (a) Compute $\mathbb{E}\left[\left(t - \sum_{k=0}^{2^n-1} \Delta_{n,k}^2\right)^2\right]$.

(b) Use the Borel-Cantelli lemma to show that

$$\sum_{k=0}^{2^n-1} \Delta_{n,k}^2 \xrightarrow[n\to\infty]{\text{a.s}} t$$

[Hint: Use the computation from question (a) together with Chebychev's inequality]