Problem set # 6

Only turn in the questions or problems marked with a (♣) for grading. It is however recommended that you try as many questions as you can.

1. **Poissonization of the multinomial (♣).**

   Let $(\xi_n)_{n\geq 0}$ be a sequence of iid random variables with values in $\{1,\dots,m\}$ and distribution
   $$\mathbb{P}(\xi_n = k) = p_k, \quad \text{for } 1 \leq k \leq m.$$
   Let N be a random variable independent from the sequence (ξ_n) with Poisson(λ) distribution i.e.
   $$\mathbb{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!}, \quad \text{for } n \geq 0.$$
   Finally let
   $$X_j := \sum_{n=0}^{N} 1(\xi_n = j), \quad \text{for } 1 \leq j \leq m.$$
   Show that the variables $X_1,\dots,X_m$ are mutually independent and X_j has a Poisson$(p_j\lambda)$ distribution.

2. **Poisson point process (♣).**

   Let $(S,\mathcal{S})$ be a measurable space and suppose that μ is a measure on $(S,\mathcal{S})$ (not necessarily a probability measure).

   We assume that μ finite i.e. $0 < \mu(S) < \infty$. Our aim is to construct a process $(N(A), A \in \mathcal{S})$ such that the following properties hold

   (i) For any $A \in \mathcal{S}$ the variable $N(A)$ has Poisson$(\mu(A))$ distribution.

   (ii) If $A_1,\dots,A_k$ are disjoint sets in $\mathcal{S}$, the variables $N(A_1),\dots,N(A_k)$ are disjoint.

   To do so we proceed as follows. Let (ξ_n) be a sequence of iid random variables with distribution $\frac{\mu(\cdot)}{\mu(S)}$ (i.e. the normalization of μ to a probability measure), and consider a random variable M with Poisson$(\mu(S))$ distribution independent of the sequence (ξ_n). For $A \in \mathcal{S}$ we set
   $$N(A) := \sum_{n=0}^{M} 1(\xi_n \in A).$$

   (a) Show that this process satisfies the desired conditions (i) and (ii). [Hint: Problem (1).]

Now we remove the assumption that μ is finite. We assume instead that μ is σ-finite i.e. there exists a countable sequence of disjoint set $(A_i)_{i\geq 0}$ in $\mathcal{S}$ such that

$$S = \bigsqcup_{i=0}^{\infty} S_i, \quad \text{and} \quad \mu(S_i) < \infty \quad \text{for all } i \geq 0.$$

In question (a) we have constructed Poisson processes N_i on $(S_i, \mathcal{S}_i)$ (here the σ-algebra $\mathcal{S}_i$ consists of the sets $S_i \cap A$ for $A \in \mathcal{S}$). We extend each N_i to a process on $(S, \mathcal{S})$ by setting

$$N_i(A) := N_i(A \cap S_i) \quad \text{for } A \in \mathcal{S}.$$

Now we define a process $(N(A), A \in \mathcal{S})$ as follows

$$N(A) = \sum_{i\geq 0} N_i(A).$$

(b) Show that N is a Poisson process on $(S, \mathcal{S})$ with intensity μ i.e.

- For any $A \in \mathcal{S}$ the variable $N(A)$ has Poisson$(\mu(A))$ distribution.
- If $A_1, \ldots, A_k$ are disjoint sets in $\mathcal{S}$, the variables $N(A_1), \ldots, N(A_k)$ are disjoint.

The process N we just constructed is called the Poisson process on $(S, \mathcal{S})$ with intensity measure $\mu(\cdot)$.