Problem set # 5

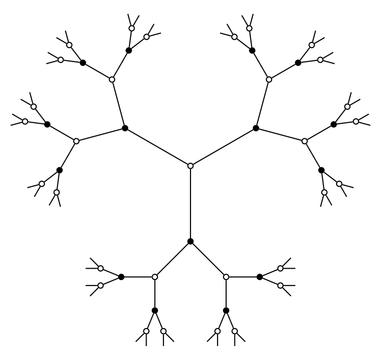
Only turn in the questions or problems marked with a (♣) for grading. It is however recommended that you try as many questions as you can.

1. **Aggregation of states (♣).**

   Let $X = (X_n)$ be a Markov chain with transition kernel P on a countable state space S. Suppose that $S = \bigsqcup_{i\in I} S_i$ is a partition of S. For any element $x \in S$ we denote by $i(x)$ the unique index in I such that $x \in S_{i(x)}$. Now define a process $Y = (Y_n)$ with values in I as follows
   $$Y_n = i(X_n), \quad \text{for } n \geq 0.$$
   Suppose that for any $i, j \in I$
   $$\sum_{z\in S_j} P(x,z) = \sum_{z\in S_j} P(y,z) \quad \text{for all } x, y \in S_i$$
   and denote this quantity by $Q(i,j)$. Show that $Y = (Y_n)_{n\geq 0}$ is a Markov chain and determine its transition kernel.

2. **Simple random walk on regular trees (♣).**

   Let T_q be a q-regular tree. This means that T_q is an infinite tree where each vertex as exactly q-neighbors. The following figure depicts a 3-regular tree.

   The tree T_q is naturally equipped with the structure of a metric space where the distance $d(v,w)$ between two vertices v, w is the smallest number of steps one needs to take to go from v to w along the edges of T_q.

   Let $S = (S_n)$ be the simple random walk on T_q, i.e. if S is at a vertex v as some time n then the chain moves equally likely to one of the q neighbors of v.

   (a) Show that this chain is transient. [Hint: pick some vertex v to start the chain (S_n) at and study $D_n = d(v, S_n)$.]

   (b) Since $S = (S_n)$ is transient one could think that S_n is moving away from its starting point v at some speed. Make this statement rigorous and determine the speed at which S_n moves away from its starting point.

3. **A Poisson jump chain.**

   Let $X = (X_n)$ be the Markov chain with state space $\mathbb{Z}$ such with the following dynamics

   - if the chain is at a state $i > 0$ then it moves to $i - 1$ with probability 1.
   - if the chain is at a state $i < 0$ then it moves to $i + 1$ with probability 1.
   - if the chain is at 0 then it stays at 0 with probability e^{-1} or moves to $i \in \mathbb{Z} \setminus \{0\}$ with probability $e^{-1}/(2|i|!)$.

   (a) Show that this chain is irreducible and aperiodic.

   (b) (♣) Determine the invariant measures of this chain. Deduce that all the states are positive recurrent.

4. **Birth and death chains.**

   Let $(r_i, p_i, q_i)_{i\in\mathbb{N}}$ be real numbers such that $p_i + r_i + q_i = 1$ for all $i = 0, 1, 2\dots$ and suppose that $q_0 = 0$. Consider the $\mathbb{N}$-valued stochastic process (X_n) such that for $i \in \mathbb{N}$:

   - $\mathbb{P}(X_{n+1} = i + 1|X_n = i) = p_i$,
   - $\mathbb{P}(X_{n+1} = i - 1|X_n = i) = q_i$,
   - $\mathbb{P}(X_{n+1} = i|X_n = i) = r_i$.

   Let $T_c = \inf\{n \geq 0\colon X_n = c\}$ for any $c \in \mathbb{N}$. We suppose that $p_i, q_i > 0$ for $i \geq 1$ and $p_0 > 0$.

   (a) Find a function $\phi\colon \mathbb{N} \to \mathbb{R}$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(X_n)$ is a martingale.

   (b) (♣) Let $a < x < b$ be elements in $\mathbb{N}$. Using the martingale you previously constructed, find $\mathbb{P}_x(T_a < T_b)$ and prove that $\mathbb{P}_x(T_0 > T_M) \geq \frac{\phi(x)}{\phi(M)}$.

   (c) Deduce a condition on p_i, q_i, r_i under which 0 is recurrent in this chain.