Problem set 4

Only submit the parts that are marked with a (♣).

1. **A sufficient criterion for recurrence of random walks on $\mathbb{Z}$.**

   The goal of this problem is to prove the following result:

   Theorem 1. *Consider the following random walk on (S_n) on $\mathbb{Z}$ such that*

   $$S_{n+1} = S_n + \xi_{n+1}, \qquad \textit{for } n \geq 0.$$

   where (ξ_i) is a sequence of i.i.d $\mathbb{Z}$-valued random variables with distribution μ. We suppose moreover that $\mathbb{E}[\xi_1] = 0$ and

   $$\mathbb{E}[|\xi_1|] = \sum_{k \in \mathbb{Z}} |k| \mu(k) < \infty.$$

   Then all the states are recurrent, moreover the chain is irreducible if and only if the group generated by $\Gamma := \{x \in \mathbb{Z} \colon \mu(x) > 0\}$ is all of $\mathbb{Z}$.

   The process (S_n) is a Markov chain on the state space $\mathbb{Z}$. Let $P = (P(x,y))_{x,y \in \mathbb{Z}}$ be its transition kernel and denote the number of visits to a state $x \in \mathbb{Z}$ by

   $$N_x := \sum_{n=0}^{\infty} 1(S_n = x).$$

   We write $\mathbb{P}_0$ when the chain (S_n) starts from $S_0 = 0$. Finally let $U = (U(x,y))_{x,y \in \mathbb{Z}}$ be the potential kernel i.e.

   $$U(x,y) := \mathbb{E}_x[N_y] = \sum_{n=0}^{\infty} P^n(x,y).$$

   (a) Express P in terms of μ and show that either all states are transient or all states are recurrent. [Hint: Show that $U(x,y)$ depends only on $x - y$.]

   Suppose now that that 0 is transient.

   (b) (♣) Show that $U(0,x) \leq U(x,x)$. [Use the strong Markov property to compute $\mathbb{E}_0[N_x]$ by conditioning on $(T_x < \infty)$ where $T_x := \inf\{n \geq 0 \colon S_n = x\}$.]

   (c) (♣) Deduce that there exists a constant $C > 0$ not depending on n such that

   $$\sum_{x \leq |n|} U(0,x) \leq Cn, \quad \text{for any } n \geq 1.$$

   (d) (♣) Show that for any $\epsilon > 0$ the exists $n_0 > 0$ such that for any $n \geq n_0$

   $$\mathbb{P}_0(|S_n| \leq \epsilon n) > \frac{1}{2}.$$

(e) Fix an $\epsilon > 0$ and n_0 as in the previous question. Show that if $n_0 \leq m \leq n$ are integers then

$$\sum_{|x| \leq \epsilon n} P^m(0,x) \geq \sum_{|x| \leq \epsilon m} P^m(0,x) > \frac{1}{2}.$$

Sum over $m \in \{n_0, \dots, n\}$ and deduce

$$\sum_{|x| \leq \epsilon n} U(0,x) \geq \sum_{m=n_0}^{n} \sum_{|x| \leq \epsilon n} P^m(0,x) > \frac{n - n_0}{2}.$$

(f) (♣) Now use question (c) to get a contradiction and deduce that 0 is recurrent and hence all states are.

(g) (♣) Show that $H := \{x \in \mathbb{Z} \colon U(0,x) > 0\}$ is a subgroup of $\mathbb{Z}$ and H is contained in the subgroup generated by Γ.

(h) Now deduce that the chain (S_n) is irreducible if and only if $H = \mathbb{Z}$.

2. **Mean exit times.** Let X be a Markov chain on a countable state space S and transition kernel P. Let $A \subset S$ such that $S \setminus A$ is finite and $T_A := \{n \geq 0 \colon X_n \in A\}$. We assume that $\mathbb{P}_x(T_A < \infty) > 0$ for any $x \in S \setminus A$. Finally, let h_A be the following function

$$h_A(x) := \mathbb{E}_x[T_A].$$

(a) (♣) Show that

$$h_A(x) = 1 + \sum_{y \in S} P(x,y) h_A(y). \tag{1}$$

(b) Argue that we actually have $\mathbb{P}_x(T_A < \infty) = 1$ for all $x \in S \setminus A$. [Hint: recall that $S \setminus A$ is finite.]

(c) (♣) Show that if a function $h \colon S \to \mathbb{R}$ satisfies equation (1) then $M_n := h(X_{n \wedge T_A}) + n \wedge T_A$ is a martingale.

(d) (♣)Use the previous question to deduce that h_A is the unique function that is 0 on A and satisfies (1)

3. **A transient Markov chain related to discrete uniform records (♣).**

Consider the Markov chain $Z_n := (X_n, Y_n)$ with state space $\mathbb{Z}^2_{\geq 0}$. The chain (Z_n) moves from a state (i,j) to $(i+1,j)$ or (i,k) for some $0 \leq k \leq j-1$, and all of these transitions are equally likely. Let $T := \inf\{n \geq 0 : Y_n = 0\}$.

Find the distribution of X_T. [Hint: Find equations that the functions $f_{i,j}(z) := \mathbb{E}_{(i,j)}[z^{X_T}]$ (for $0 \leq z \leq 1$) satisfy.]