Problem set 3

1. **A convergent series via martingales (♣).**

   Let $(\xi)_{n\geq 0}$ be a sequence of i.i.d random variables with $\mathbb{P}(\xi_n = \pm 1) = 1/2$ and let $a = (a_k)$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} a_k^2 < \infty$. Finally, let

$$S_n := \sum_{k=1}^{n} a_k \xi_k, \quad \text{for } n \geq 1.$$

   (a) Introduce a suitable martingale to show that S_n converges a.s. to an $\mathbb{R}$-valued random variable S.

   (b) Explain why the condition $\sum_{k=1}^{\infty} a_k^2 < \infty$ is crucial. [Hint: there may be an exercise in HW1 that you might want to use here.]

   (c) Show that (S_n) is uniformly integrable.

2. **Simple random walk with drift (♣).**

   Let $x \in \mathbb{Z}$ be an integer and $0 < p < 1/2$ and set $q = 1 - p$. Consider the random walk

$$S_0 = x \quad \text{and} \quad S_n = x + \xi_1 + \cdots + \xi_n,$$

   where $(\xi)_{n\geq 0}$ is a sequence of i.i.d random variables with

$$\mathbb{P}(\xi_n = 1) = p \quad \text{and} \quad \mathbb{P}(\xi_n = -1) = q.$$

   For any $y \in \mathbb{Z}$ write $T_y := \inf\{n \geq 0 \colon S_n = y\}$.

   (a) Let a, b be integers such that $a \leq x \leq b$. Using a suitable martingale, find $\mathbb{P}(T_a < T_b)$.

   (b) Check that $T_b \geq b - x$ and $T_a \geq x - a$ a.s. then deduce $\mathbb{P}(T_a < \infty)$ and $\mathbb{P}(T_b < \infty)$.

   (c) Using a geometric martingale of the form $Z_n = r^n \rho^{S_n}$, find the law of T_a. [Hint: One way to encode a probability distribution on $\mathbb{N}$ is through the probability generating function.]

   (d) Find the law of T_b too. [Hint: Be mindful that $\mathbb{P}(T_b = \infty) > 0$ here.]