Problem set # 2

1. **Branching processes.** Let ν be a probability distribution on $\mathbb{N} := \mathbb{Z}_{\geq 0}$ and denote by $m \in (0, \infty]$ the mean of ν. Let $(\xi_{n,k})_{n,k\geq 0}$ be an i.i.d sequence of random variables with distribution ν and $(Z_n)_{n\geq 0}$ be the stochastic process defined as follows

$$Z_0 = 1, \quad \text{and} \quad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n,k}$$

This is a branching process with offspring distribution ν.

(a) Suppose ξ is a ν-distributed random variable. Show that the probability generating function $\phi(x) := \mathbb{E}[x^{\xi}]$ defined for $0 \leq x \leq 1$ is continuous non-decreasing and convex on $[0, 1]$ and continuously differentiable on $(0, 1)$.

(b) Let $\phi_n(x) := \mathbb{E}[x^{Z_n}]$. Find an expression for ϕ_n in terms of ϕ.

(c) Define the extinction probability of (Z_n) as $p_e := \mathbb{P}[Z_n = 0 \text{ for some } n \geq 0]$. Show that $\phi(p_e) = p_e$.

(d) Show that if $m \leq 1$ and $\nu(\{1\}) < 1$ then $p_e = 1$.

(e) (♣) Assume that $m > 0$. Show that then $W_n := Z_n/m^n$ converges almost surely to a real valued random variable W. Recover, independently of question (d), that if $m < 1$ then Z_n goes extinct with probability 1.

(f) (♣) Determine all the probability distributions ν on $\mathbb{N}$ with mean 1 such that $(Z_n)_{n\geq 0}$ is uniformly integrable. [Hint: question (d).]

2. **Uniform integrability (♣).**

(a) Show that a family (X_i) is uniformly integrable if and only if

$$\lim_{\delta \to 0, \delta > 0} \sup_{\substack{i \in I \\ A \in \mathcal{F},\ \mathbb{P}(A) < \delta}} \mathbb{E}\left[|X_i| 1_A\right] = 0.$$

(b) Deduce that if $(X_n)_{n\geq 0}$ is a uniformly integrable then the family $(Z_{n,\mathcal{G}} := \mathbb{E}[X_n|\mathcal{G}])_{n,\mathcal{G}}$ is uniformly integrable?

3. **Wald's identity (♣).**

Let (ξ_n) be an i.i.d sequence of random variables in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mu := \mathbb{E}[\xi_1]$, and set $\mathcal{F}_n := \sigma(\xi_1, \ldots, x_n)$ to be the natural filtration of (ξ_n).

(a) Suppose that T is a stopping time such that $\mathbb{E}[T] < 0$. Show that $S_{n\wedge T} - \mu(n \wedge T)$ is uniformly integrable. [Hint: Consider the random variables $Z_n := \sum_{i=1}^{n} |\xi_i - \mu| 1_{T \geq i}$.]

(b) Deduce that $S_T \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}[S_T] = \mu\mathbb{E}[T]$

4. **Lipschitz functions on $[0,1]$.**

In this problem, we want to prove the following result

Theorem 1. *Let $f:[0,1]\to\mathbb{R}$ be a Lipschitz function, i.e. there exists $C>0$ such that $|f(x)-f(y)|\leq C|x-y|$ for any $x,y\in[0,1]$. Then there exists a bounded Borel-measurable function $g:[0,1]\to\mathbb{R}$ such that*

$$f(x)=f(0)+\int_0^x g(u)du.$$

To show this result we can proceed as follows. Let U be $[0,1)$-valued random varialbe with the uniform distribution and set

$$X_n := 2^{-n}\lfloor 2^n U\rfloor \quad \text{and} \quad Z_n := 2^n(f(X_n+2^{-n})-f(X_n)),$$

and $\mathcal{F}_n := \sigma(X_0,\dots,X_n)$ the natural filtration of $(X_n)_{n\geq 0}$.

(a) Check that $\sigma(X)=\mathcal{F}_\infty := \sigma(X_0,X_1,\dots)$ and that $\mathcal{F}_n=\sigma(X_n)$.

(b) Show that (Z_n) is a bounded martingale with respect to $(\mathcal{F}_n)$

(c) Show that there exists a bounded Borel-measurable function $g:[0,1]\to\mathbb{R}$ such that $Z_n\to g(X)$ a.s. as $n\to\infty$.

(d) Check that a.s. for any $n\geq 0$:

$$Z_n = 2^n\int_{X_n}^{X_n+2^{-n}} g(u)du.$$

(e) Conclude that for any $x\in[0,1]$

$$f(x)=f(0)+\int_0^x g(u)du.$$

5. **Distribution of a martingale limit.**

Let $a\in(0,1)$ and consider the stochastic process $(X_n)_{n\geq 0}$ with values in $[0,1]$ such that $X_0=a$ and for any $n\geq 0$

$$\mathbb{P}\left(X_{n+1}=\frac{X_n}{2}|\mathcal{F}_n\right)=1-X_n, \quad \text{qnd} \quad \mathbb{P}\left(X_{n+1}=\frac{1+X_n}{2}|\mathcal{F}_n\right)=X_n,$$

where $\mathcal{F}_n := \sigma(X_0,\dots,X_n)$.

(a) Show that X_n converges a.s. to a random variable $X\in[0,1]$.

(b) Prove that $\mathbb{E}[(X_{n+1}-X_n))^2]=\frac{1}{4}\mathbb{E}(X_n(1-X_n))$

(c) (♣) Find the distribution of Z. [Hint: You may want to send n to ∞ in the last question and compute the moments of Z using L^p convergence of martingales. Running a simulation is also not a bad idea.]