Problem set # 1

Only turn in the questions or problems with a (♣) next to them for grading. It is however recommended that you try as many questions as you can.

1. **Conditional expectation ♣.**

   Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$. Show that if $Z = \mathbb{E}[X|\mathcal{G}]$ and X have the same law then $Z = X$ a.s.

   [Hints: you may want to consider the case where X is in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ first. Another approach is:

   (a) Start by checking that $\mathbb{E}[X(1_{X>0} - 1_{Z>0})] = 0$.

   (b) Notice $1_{X>0} - 1_{Z>0} = 1_{X>0,Z\leq 0} - 1_{X\leq 0,Z>0}$

   (c) Then deduce that $\mathbb{P}(X > 0, Z \leq 0) = \mathbb{P}(X < 0, Z > 0) = 0$.

   (d) Apply what you've done so far to $X + c$ and deduce $\mathbb{P}(X > c, Z \leq c) = \mathbb{P}(X < c, Z > c) = 0$ for any $c \in \mathbb{R}$.

   (e) Conclude the desired result.

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2. **Martingales and stopping time.**

   Given an integrable stochastic process X_n adapted to a filtration $\mathcal{F}_n$, show that $(X_n, \mathcal{F}_n)$ is a martingale if and only if $\mathbb{E}[X_n|\mathcal{F}_\tau] = X_\tau$ for any non-random, finite n and all $\mathcal{F}_n$-stopping times $\tau \leq n$ where $\mathcal{F}_\tau := \{A \in \sigma(\cup \mathcal{F}_n) : A \cap \{\tau \leq i\} \in \mathcal{F}_i \ \forall i\}$. As a part of the exercise prove that $\mathcal{F}_\tau$ is a sigma-algebra for every stopping time τ.

3. **Almost sure convergence of martingales.**

   (a) Let (X_n) be a submartingale such that $\sup X_n < \infty$ a.s. and let $\xi_n := X_n - X_{n-1}$ and suppose that $\mathbb{E}(\sup \xi_n^+) < \infty$. Show that X_n converges a.s.

   [Hint: For an integer $K \geq 1$, consider the stopping time $T_K := \inf\{n \geq 0 \colon X_n > K\}$ and use the almost sure convergence theorem on $X_{n\wedge T_K}$. Then note that on the event $(T_K = \infty)$ we have $X_{n\wedge T_K} = X_n$. Then show that $\mathbb{P}(\cup_{K>0}(T_K = \infty)) = 1$.]

   (b) Let X_n, Y_n be nonnegative, integrable and $\mathcal{F}_n$-adapted. Suppose that

   $$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq (1 + Y_n)X_n \quad \text{and} \quad \sum_n Y_n < \infty \text{ a.s.}$$

   Prove that X_n converges a.s.

   (c) ♣ Suppose now that

   $$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq Y_n + X_n \quad \text{and} \quad \sum_n Y_n < \infty \text{ a.s.}$$

Prove again that X_n converges a.s.

[Hint: Construct a supermartingale Z_n from X_n and the Y_i's. Then, as in question (a), stop this martingale at a suitable stopping time to make it bounded from below then apply the almost sure convergence theorem.]

4. **Probability of bankrupcy in finite time.**

Let $x > 0$ and (X_n) i.i.d random variables with $\mathbb{E}[e^{\lambda X_k}] < \infty$ for any $\lambda > 0$. Let $c > 0$ and consider the process

$$S_0 = x \quad \text{and } S_{n+1} = S_n + (c - X_{n+1}) \text{ for } n \geq 0.$$

Finally let $T = \inf\{n \geq 0 : S_n < 0\}$.

(a) Check that $\mathbb{P}(\tau < \infty) = 1$ when $\mathbb{E}[X] > c$.

(b) ♣ From now on suppose that $\mathbb{E}X_1 < c$ and $\mathbb{P}(X_1 > c) > 0$. Show that there exists a unique $\lambda_0 > 0$ such that $V_n := e^{-\lambda_0 S_n + \lambda_0 x}$ is a martingale.

(c) ♣ Use V_n to show that $\mathbb{P}(T < \infty) \leq e^{-\lambda_0 x}$.

5. **Kolmogorov inequality.**

Let $x > 0$ (X_n) be independent, centered and square integrable random variables. Set $\sigma_n^2 = \text{Var} X_n$ and $S_n = X_1 + \cdots + X_n$.

(a) Show that for any $x > 0$ and integer $n \geq 1$ we have

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\right) \leq \text{Var}(S_n)/x^2.$$

(b) (♣) Now suppose that there exists $c > 0$ such that $|X_n| \leq c$ a.s for any n. Check that $S_n^2 - \text{Var}(S_n)$ is a martingale and deduce that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \leq x\right) \leq (x + c)^2/\text{Var}(S_n), \quad \text{for all } x > 0 \text{ and } n \in \mathbb{Z}_{>0}.$$

6. **Kolmogorov three series theorem.**

Let ξ_n be a sequence of independent real random variables. For $c > 0$ we write $\xi_n^{(c)} = \xi_n 1(|\xi_n| \leq c)$. The aim of this problem is to show that the following are equivalent

(i) $\sum_{k=0}^n \xi_k$ converges almost surely in $\mathbb{R}$.

(ii) For any $c > 0$ the two series $\sum_{n \geq 0} \mathbb{P}(|\xi_n| > c)$ and $\sum_{n \geq 0} \text{Var}(\xi_n^{(c)})$ are summable and $\sum_{k=1}^n \mathbb{E}[\xi_n^{(c)}]$ converges.

To do so proceed as follows:

(a) Assume (ii). Show that $S_n = \sum_{k=0}^n \xi_k^{(c)} - \mathbb{E}[\xi_k^{(c)}]$ is a martingale for a simple filtration to be chosen. Moreover

$$\sup_{n\geq 0}(\mathbb{E}S_n^2) < \infty.$$

Using the L^2-convergence theorem deduce that S_n converges almost surely and in L^2.

(b) ♣ Use the previous question to show that (ii) implies (i).

From now on suppose (i).

(c) Using the Borel-Cantelli lemma show that

$$\sum_{n\geq 0} \mathbb{P}(|\xi_n| > c) < \infty \quad \text{ and } \sum_{k=1}^n \xi_k^{(c)} \text{ converges a.s. in } \mathbb{R}.$$

(d) Suppose given two independent sequences (χ_n) and (χ_n^*) of the same distribution as $(\xi_n^{(c)})$ and set $Z_n = \chi_n - \chi_n^*$. Show that $\sum_{k=1}^n Z_k$ converges a.s in $\mathbb{R}$.

(e) (♣) Check that $|Z_n| \leq 2c$, $\mathbb{E}Z_n = 0$ and $\mathrm{Var}(Z_n) = 2\mathrm{Var}(\xi_n^{(c)})$. Set $S_n^* := Z_1 + \cdots + Z_n$ and check that

$$\mathrm{Var}(S_n^*) = 2\sum_{k=1}^n \mathrm{Var}(\xi_k^{(c)}),$$

and use the previous problem to deduce that there exist $\eta, x_0 > 0$ such that for any $n \geq 1$

$$\eta < \mathbb{P}\left(\max_{1\leq k\leq n} |S_k^*| \leq x_0\right) \leq \frac{(x_0+2c)^2}{\mathrm{Var}(S_n^*)}.$$

Deduce that $\sum_{n\geq 1} \mathrm{Var}(\xi_n^{(c)}) < \infty$.

(f) (♣) Recall that $S_n = \sum_{k=1}^n \xi_k^{(c)} - \mathbb{E}[\xi_k^{(c)}]$. Check that S_n is a martingale and $\sup_n \mathbb{E}[S_n^2] < \infty$ so S_n converges a.s. Show that this implies that $\sum_{n\geq 1}\mathbb{E}[\xi_n^{(c)}]$ is a convergent series.

This finishes the proof that (i) implies (ii).