LOCAL CLASS FIELD THEORY VIA LUBIN TATE FORMAL GROUPS

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Abstract. The main aim of this paper is to present the main arguments to establish local class field theory via Lubin-Tate formal groups.

1. Introduction and motivation

The goal of local class field theory is to classify all finite abelian extensions of a given local field K. In this paper we focus on the non-archimedian case as ou main concern and rather than dealing with abelian Galois extensions L/K individually we deal with them globally as subextensions of the maximal abelian extension K^{ab} (which we will define later) that sits in some separable closure of K. Our main objective it to gives the steps to prove the following

Theorem 1.1. *If K is a local field, there exists a unique group homomorphism* $\mathrm{Art} : K^\times \to \mathrm{Gal}(K^{ab}/K)$ *satisfying the two following properties*

(i) If π is a uniformizer of K and L/K is a finite unramified extension of K then we have $\mathrm{Art}(\pi)_{|L} = \mathrm{Frob}_{L/K}$

(ii) If L/K is finite abelian, Art *induces an group isomorphism* $K^\times/N(L^\times) \to \mathrm{Gal}(L/K)$ *via th map*

$$\begin{aligned} K^\times &\to \mathrm{Gal}(L/K) \\ x &\mapsto \mathrm{Art}(x)_{|L} \end{aligned}$$

Theorem 1.1 is often refered to as *Local Artin Reciprocity* and it allows us to understand and classify abelian extensions of K.

Studying abelian extensions of local fields is motivated by questions of number theoretic nature. For instance when F is a number field (finite extension of $\mathbb{Q}$) there exists a canonical everywhere unramified extension E/F such that the only primes of F that split in L are the principal primes and such that $\mathrm{Gal}(E/F)$ is isomorphic to the ideal class group of F which is a finite abelian group. Thus studying ideal class groups leads naturally to the study of finite abelian extension of F. It turns out however that studying abelian extensions of number fields is quite challenging and that it is easier to study their behavior when localized at a prime ideal $\mathfrak{p}$ which leads us to studying local fields and their abelian extensions. Classifying Galois extensions is a very ambitious endeavor and with the Langlands program there is now research being done on non-abelian class field theory.

In this paper we go throught the main steps to prove *Artin's Local Reciprocity* using Lubin-Tate formal groups and it is organized as follows. In section 2 we review some important results concerning extensions of local fields. Section 3.1 introduces formal groups and section 3.2 defines Lubin-Tate formal groups. In section 4 we build totally ramified abelian extensions of local fields and section 5 is dedicated to proving the existance and uniqueness of the Artin map Art. We warn the reader that we will be skipping some details when proofs get too complicated but we will nevertheless refer to other sources for details.

The results we discuss throughout this paper are standard results in number theory and once can find more details in many sources like [Mil], [II86] , [Ser13] and also the original Lubin-Tate paper [LT65]. Alternative proofs that use Galois cohomology are given by Milne [Mil] and Serre [Ser13] and the advantage that Lubin-Tate theory offers is a more explicit construction that avoids cohomology arguments.

2. Extensions of local lields

Let K be a non-archimedian local field with ring of integers $\mathcal{O}_K$ and $\mathfrak{m}_K$ the unique maximal ideal of $\mathcal{O}_K$. We define the residue field $k := \mathcal{O}_K/\mathfrak{m}_K \cong \mathbb{F}_q$ where q is a power of some prime number $p := \mathrm{char}(k) > 0$. If L is a finite separable extension of K we denote by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L which is also a DVR and we denote $\mathfrak{m}_L$ its unique maximal ideal and $k_L := \mathcal{O}_L/\mathfrak{m}_L$ the residue field. We denote by $e_{L/K}$ the ramification index of $\mathfrak{m}_K$ in L and $f_{L/K}$ its inertia degree. The trace and norm map are respectively denoted $Tr_{L/K}$ and $N_{L/K}$ or simply Tr, N when there is no ambiguity. If v_K, v_L are the additive valuations on K, L respectively we have $v_L(x) := \frac{1}{e_{L/K}} v_K(N_{L/K}(x))$ for any $x \in L$. We denote by K^{al} an algebraic closure of K and $K^s \subset K^{al}$ the separable closure of K in K^{al}. The valuation v_K extends uniquely to a valuation v on K^{al} which defines a norm (multiplicative valuation $|.|$) on K^{al}. For $n \geq 1$ we denote $\boldsymbol{\mu}_n$ the set of n^{th}-roots of the unit in K^{al}, notice that if $\gcd(n, p) = 1$ then $\boldsymbol{\mu}_n \subset K^s$ (we don't really need this condition in the mixed characteristic case $\mathrm{char}(K) \neq 0$). We start with the first well understood example of abelian local field extensions.

Lemma 2.1. *Let L/K be a finite Galois extension, then L/K is unramified if an only if* $\mathrm{Gal}(L/K)$ *and* $\mathrm{Gal}(k_L/k)$ *are canonically isomorphic.*

Proof. Let's prove first that k_L/k is Galois. Let $k_L = k[\alpha]$ and $\overline{f} = \mathrm{Min}_{\alpha,k}(X)$ the minimal polynomial of α. Since k is finite it is perfect and thus $\overline{f}$ has simple roots. Let $f \in \mathcal{O}_K[X]$ be a lift of $\overline{f}$, by Hensel's lemma there exists $a \in L$ such that $f(a) = 0$ and $a \equiv \alpha \mod \mathfrak{m}_L$. Since L is a Galois extension the polynomial f splits in L and thus $\overline{f}$ splits in k_L which means that k_L/k is a Galois extension. The action of $\mathrm{Gal}(L/K)$ preserves the valuation v_L so there exists a canonical homomorphism $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k), \sigma \mapsto \overline{\sigma}$.

Now when L/K is unramified we have $[L : K] = [k_L : k]$ so we have $L = K[a]$. The automorphisms in $\sigma \in \mathrm{Gal}(L/K)$ map a into distinct conjugates. Thus the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k)$ is a surjective group homomorphism and since the two groups have the same cardinality it is an isomorphism. Conversely if the two Galois groups are isomorphic then we have $f_{L/K} := [k_L : k] = |\mathrm{Gal}(k_L/k)| = |\mathrm{Gal}(L/K)| = [L : K]$ then the extension L/K is unarmified. ☐

Proposition 2.2. *For any positive integer n, $K_n := K(\boldsymbol{\mu}_{q^n-1})$ is the unique unramified extension of K of degree n and it is a Galois extension with a cyclic Galois group.*

Proof. If L/K is a degree n unramified extension then we have $[k_L : k] = n$ which means that $k_L \cong \mathbb{F}_{q^n}$. Then since $\boldsymbol{\mu}_{q^n-1}(k) \subset k_L$ we also have by Hensel's lemma $\boldsymbol{\mu}_{\mathbf{q^n-1}}(K) \subset L$ which means that $K_n \subset L$. Now it suffices to show that K_n is indeed a degree n unramified extension which implies showing that $[K_n : K] = n$ and this we can do again using Hensel's lemma. □

Since the composite field of two unramified extensions of K is also a ramified extension, there exists then a canonical maximal unramified extension of K which we denote by $K^{ur} := \cup_n K_n$. From lemma 2.1 and proposition 2.2 we have $\mathrm{Gal}(K_n/K) \cong \mathrm{Gal}(\mathbb{F}_{q^n}/F_q) = \mathbb{Z}/n\mathbb{Z}$. We then deduce that $\mathrm{Gal}(K^{ur}/K) \cong \varprojlim_n (Z/n\mathbb{Z}) = \widehat{\mathbb{Z}} \cong \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The Frobenius map $\mathrm{Frob} \in \mathrm{Gal}(K^{ur}/K)$ is the automorphism that reduces modulo $\mathfrak{m}_K$ to the q^{th}-power map on $\overline{\mathbb{F}}_q$. Galois theory tells us the the composite of two abelian extensions is also abelian then we can also define a canonical maximal abelian extension of K which we denote K^{ab}.

3. Lubin-Tate formal groups

3.1. Formal groups.

For non-zero commutative ring A we denote by $A[T]$ the ring of formal power series in one variable T and for an integer $n \geq 1$ we denote by $A[[X_1, \dots, X_n]]$ the ring of formal power series in n variables. When $f \in TA[T]$ and $g \in A[T]$ one can make sens of the composition law $g \circ f(T) := g(f(T))$ because we can recursively define the coefficients (all coefficient are finite sum and we can make sens of finite sums on A). When f has a non-zero constant term computing the constant term of $g(f(X))$ involves dealing with infinite sums in A. Before defining formal group laws over A we start with the following useful lemma.

Lemma 3.1. *If $f = a_1T + a_2T^2 \cdots \in TA[[T]]$ then $a_1 \in A^\times$ if and only if there exists $g \in TA[[T]]$ such that $g \circ f = f \circ g = T$. In this case g is unique and we denote it f^{-1} which we call the inverse of f.*

Proof. For $f = \sum_{i\geq 1} a_iT^i, g = \sum_{i\geq 1} b_iT^i \in TA[[T]]$, let $(c_i)_{i\geq 1}$ be the sequence in A such that $f \circ g = \sum_{i\geq 1} c_iT^i$. Then we have $c_1 = a_1b_1$, $c_2 = a_1b_2 + a_2b_1^2$ and in general we have $c_n = a_1b_n + P_n(a_1, \dots, a_n, b_1, \dots, b_{n-1})$ where P_n is some polynomial. If f^{-1} exists is is clear from the form of the coefficient $c_1 = a_1b_1 = 1$ that $a_1 \in A^\times$. Conversely if $a_1 \in A^\times$ we can iteratively define a unique sequence (b_i) to get $c_1 = 1$ and $c_i = 0$ for $i \geq 2$ which means that f^{-1} exists. □

We now define commutative formal group laws over A which will be useful for later constructions.

Definition 3.2. A commutative formal group over A is a formal power series of two variables $F(X, Y) \in A[[X, Y]]$ satisfies the following axioms :

$$i) \quad F(X,Y) = F(Y,X) \quad (commutativity)$$
$$ii) \quad F(F(X,Y),Z) = F(X,F(Y,Z)) \quad (associtivity)$$
$$iii) \quad F(X,Y) \equiv X+Y \mod (\deg 2)$$

Condition (iii) in the above definition can actually be replaced with the condition $F(X,0) = X$ and $F(0,Y) = Y$. In fact if $f(X) = F(X,0)$ then we have $f(X) \equiv X \mod (\deg 2)$ then by Lemma 3.1 f has an inverse f^{-1}. Then using (ii) we have $F(F(X,0),0) = F(X,F(0,0))$ which translates to $f \circ f = f$ and by composing with f^{-1} we get $f(X) := F(X,0) = X$ and the same argument applies to $F(0,Y) = Y$. This condition is actually very important as we will see in the following lemma

Lemma 3.3. *For any power series $f \in TA[[T]]$ there exists $g \in TA[[T]]example$ such that $F(f(T),g(T)) = 0$.*

Proof. It suffice to prove the lemma for $f = T$. In fact if there exists $h(T)\ inTA[T]$ such that $F(T,h(T)) = 0$ then for any $f \in TA[T]$ we have $F(f(T),h(f(T))) = 0$. Now we prove the result for $f(T) = T$. From the discussion above we can write $g(T) := F = X + Y + \sum_{i \geq 1} a_{i,j} X^i Y^j$ then we get the expression $F(T,h(T)) = T + h(T) + \sum_{i \geq 1} a_{i,j} T^i h(T)^j$. Then by writing $h(T) := \sum_{i \geq 1} b_i T^i$ we get $g(T) \equiv (1+b_1)T \mod (\deg 2)$, $g(T) \equiv (1+b_1)T + (b_2 + a_{1,1})T^2 \mod (\deg 3)$ and in general we have

$$g(T) \equiv \sum_{i=1}^{n} (b_i + q_i)T^i \mod (\deg(n+1))$$

where q_i is some polynomial expression in $b_1, \ldots, b_{i-1}$ for $i \geq 2$. Then we can clearly define the sequence (b_i) recursively in such a way that $g(T) = F(T,h(T)) = 0$. ☐

We can then equip the ideal $(T) := TA[[T]]$ with a new abelian group structure by defining the law $f +_F g = F(f,g)$. In fact by definition of F this law is commutative, associative and by the lemma above every element $f \in (T)$ has an inverse $i_F(f)$ which is then unique.

Definition 3.4. Let F, G be two commutative formal groups over A and $f \in (T)$ a power series with no constant coefficient. We call f a homomorphism from F to G and we write $f : F \to G$ if it satisfies

$$f(F(X,Y)) = G(f(X), f(Y)) \quad i.e \quad f(X +_F Y) = f(X) +_G f(Y)$$

We denote by $\mathrm{Hom}(F,G)$ the set of homomorphisms from F to G. If $f \in \mathrm{Hom}(F,G)$ has an inverse it is called an isomorphism.

Lemma 3.5. *The set $\mathrm{Hom}(F,G) \subset (T)$ is a subgroup of (T) for the addition $+_G$ and $(\mathrm{End}(F) := \mathrm{Hom}(F,F), +_F, \circ)$ is a ring.*

Proof. We have $T \in \mathrm{Hom}(F,G) \neq \emptyset$ and for $f, g \in \mathrm{Hom}(F,G)$ and $h = f +_G g$ we have $h(F(X,Y)) = G(f,g)(F(X,Y)) = G(f(F), g(F))$. Then since $f, g : F \to G$ we have $h(F(X,Y)) = G(G(f(X), f(Y)), G(g(X), g(Y)))$ then we deduce that $h(F(X,Y)) = (f(X) +_G f(Y)) +_G (g(X) +_G g(Y))$. Then since $+_G$ is by definition associative and commutative we deduce that $h(F(X,Y)) = (f(X) +_G g(X)) +_G (f(Y) +_G g(Y))$ which means that $h(X +_F Y) = h(X) +_G h(Y)$. Then $\mathrm{Hom}(F,G)$ is indeed a subgroup of

$((T), +_G)$. For the second part we clearly see that $\mathrm{End}(F)$ is closed under composition and it's not so hard to check the ring axioms. □

Example 3.6. Let $\rho : \mathfrak{m}_K \to 1 + \mathfrak{m}_K, x \mapsto 1 + x$. The power series $F(X,Y) = X + Y + XY$ is a formal group law and endowing the maximal ideal $\mathfrak{m}_K$ with the group $+_F$ makes the map $\rho : (\mathfrak{m}_K, +_F) \to (1 + \mathfrak{m}_K, \times)$ a group isomorphism.

3.2. **Lubin-Tate Formal groups.**

Now that we have established some preliminaries on commutative formal group laws we go back to the local field settings in section 2 to introduce Lubin-Tate formal groups. We recall that K is a local field with ring of integers $\mathcal{O}_K$ whose maximal ideal is $\mathfrak{m}_K$ and $\mathcal{O}_K/\mathfrak{m}_K \cong \mathbb{F}_q$. For any uniformizer π of K we define the set $\mathcal{P}(\pi)$ of power series $f \in \mathcal{O}_K[[X]]$ satisfying

$$i) \quad f(X) \equiv \pi X \mod (\deg 2) \quad \text{and} \quad (ii) \quad f(X) \equiv X^q \mod (\mathfrak{m}_K)$$

In other words $\mathcal{P}(\pi)$ is the set of power series in (X) with linear coefficient π and that reduce modulo π to the Frobenius map. We start with the following important lemma:

Lemma 3.7. *Let $f, g \in \mathcal{P}(\pi)$ and $\phi(X_1, \ldots, X_n)$ a linear form with coefficients in $\mathcal{O}_K$. There exists a unique power series $\Phi \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ such that*

$$i) \quad \Phi \equiv \phi \mod \deg 2$$
$$ii) \quad f(\Phi(X_1, \ldots, X_n)) = \Phi(g(X_1), \ldots, g(X_n))$$

Proof. Similarly to the proof of Lemma 3.3 we construct Φ recursively. Let $\Phi \in \mathcal{O}_K[[X_1 =, \ldots, X_n]]$ we write $\Phi = \sum_{k \geq 1} \phi_k$ where ϕ_k is a homogeneous polynomial of degree k and we define the partial sums $\Phi_r = \sum_{k=1}^{r} \phi_k$. We have $f \circ \Phi \equiv f \circ \Phi_r \mod (\deg r + 1)$ and $\Phi \circ g := \Phi(g(X_1), \ldots, g(X_n)) \equiv \Phi_r \circ g \mod (\deg 2)$. Now we choose $\phi_1 := \phi$ and we then have $\Phi \equiv \phi \mod (\deg 2)$ and thus Φ satisfies the first condition. Since $f, g \in \mathcal{P}(\pi)$ and we also have $f \circ \phi_1 \equiv \pi \phi_1 \mod (\deg 2) \equiv \phi_1 \circ g$. Now suppose that we have picked $\phi_1, \ldots, \phi_r$ in such a way that $f \circ \Phi_r \equiv \Phi_r \circ g \mod (\deg r+1)$ for some $r \geq 1$. We have $\Phi_{r+1} \circ g = \Phi_r \circ g + \phi_{r+1} \circ g \equiv \Phi_r \circ g + \pi^{r+1} \phi_{r+1} \mod (\deg r + 2)$ and also $f \circ \Phi_{r+1} \equiv f \circ \Phi_r + \pi \phi_{r+1} \mod (\deg r + 2)$. Then to pick the term ϕ_{r+1} we need to solve the equation $(\pi - \pi^{r+1}) \phi_{r+1} \equiv \Phi_r \circ g - f \circ \Phi_r \mod (\deg r + 2)$. Since we already have $f \circ \Phi_r \equiv \Phi_r \circ g \mod (\deg r + 1)$ we need only solve $(\pi - \pi^{r+1}) \phi_{r+1} = P_{r+1}$ where P_{r+1} is the homogeneous terms with degree $r + 1$ of the series $\Phi_r \circ g - f \circ \Phi_r$. Then it only remains to prove that $P_{r+1} \mod (\pi) = 0$ and for that we have $P_{r+1} \equiv \Phi_r \circ (g \mod (\pi)) - (f \mod \pi) \circ \Phi_r \equiv \Phi_r(X_1^q, \ldots, X_n^q) - \Phi_r(X_1, \ldots, X_n)^q$, and since we now are working with power series over $k \cong \mathbb{F}_q$ the last quantity is equals 0. Thus ϕ_{r+1} is uniquely determined from the choice of $\phi_1, \ldots, \phi_r$. This construction proves existence and uniqueness because we had only one choice for ϕ_1 which completely determines all the other terms. □

Proposition 3.8. *For any $f \in \mathcal{P}(\pi)$ there exists a unique formal commutative formal group law $F_f \in \mathcal{O}_K[[X, Y]]$ such that $f \in \mathrm{End}(F_f)$.*

Proof. By applying lemma 3.7 to $f = g$ and $\phi(X, Y) = X + Y$ there exists a unique power series $F_f(X, Y)$ such that $f \circ F_f = F_f \circ f$. It then remains to show that F_f is indeed a commutative formal group over $\mathcal{O}_K$. We have by definition $F_f \equiv X + Y$

mod (deg 2) so the last condition of definition 3.2 is satisfied. Now let $\Phi(X,Y,Z) := F_f(X, F_f(Y,Z))$ and $\Psi(X,Y,Z) := F_f(X, F_f(Y,Z))$ we clearly have $\Phi(X,Y,Z) \equiv \Psi(X,Y,Z) \mod \deg 2$ and since $f \circ \mathbb{F}_f = \mathbb{F}_f \circ f$ we also have $f \circ \Phi = \Phi \circ f$ and $f \circ \Psi = \Psi \circ f$ thus using the uniqueness in 3.7 we deduce that $\Phi = \Psi$ which meas that F_f is associative. We prove the commutativity of F_f similarly. □

Let $f, g \in \mathcal{P}(\pi)$, by applying Lemma 3.7 to $\phi(X) = \alpha X$ for any $\alpha \in \mathcal{O}_K$, there exists a unique power series $[\alpha]_{f,g} \in \mathcal{O}_K[[X]]$ such that

$$i) \quad [\alpha]_{f,g} \equiv \alpha X \mod \deg 2$$
$$ii) \quad f \circ [\alpha]_{f,g} = [\alpha]_{f,g} \circ g$$

when $f = g$ we simply write $[\alpha]_f$ instead of $[\alpha]_{f,f}$

Lemma 3.9. *The map $\mathcal{O}_K \xrightarrow{[.]_f} \mathrm{End}(F_f)$ is well defined, injective and is a ring homomorphism from $(\mathcal{O}_K, +, \times)$ to $(\mathrm{End}(F_f), +_{F_f}, \circ)$.*

Proof. For $\alpha \in \mathcal{O}_K$ we have $[\alpha]_f \circ F_f \equiv \alpha(X+Y) \mod (\deg 2) \equiv F_f \circ [\alpha]_f$ and since $f \in \mathrm{End}(F_f)$ both series commute with f then using the uniqueness in lemma 3.7 we deduce that $[\alpha]_f \in \mathrm{End}(F_f)$ then $[.]_f$ is indeed well defined. The injectivity stems from the fact that we can recover α using $[\alpha]_f \equiv \alpha X \mod (\deg 2)$. We check the ring homomorphism conditions by once more using uniqueness in lemma 3.7 □

For any $f \in \mathcal{O}_K[[X]]$ the powers series F_f defines a commutative formal group law and the map $[.]_f : \mathcal{O}_K \, \mathrm{End}(F_f)$ define a formal scalar multiplication. For example when L/K is a finite extension $(\mathfrak{m}_L, +_{F_f})$ is an abelian group and we can endow it with an $\mathcal{O}_K$-module structure where we define the scalar multiplication $a.x$ for $a \in \mathcal{O}_K, x \in \mathfrak{m}_L$ by $a.x := [a]_f(x) \in \mathfrak{m}_L$ the latter is an element in $\mathfrak{m}$ because any power series in $x \in \mathfrak{m}_L$ converges. This module structure will turn out to be very useful in section 4 to build some interesting abelian extensions of K. A legitimate question to ask is how much does this structure depend on the choice of the uniformizer π and the power series $f \in \mathcal{P}(\pi)$? The choice of the uniformizer is very important as we shall see in the the next section when we construct abelian extensions. It turns out however, that the choice of $f \in \mathcal{P}(\pi)$ is unimportant as we explain in the following proposition.

Proposition 3.10. *Let $f, g \in \mathcal{P}(\pi)$ and $a \in \mathcal{O}_K$ the formal $\mathcal{O}_K$-modules structures $(F_f, [.]_f), (F_g, [.]_g)$ are isomorphic.*

Proof. For $f, g, h \in \mathcal{P}(\pi)$ and $a, b \in \mathcal{O}_K$ we have $[ab]_{f,h} = [a]_{f,g} \circ [b]_{g,h}$ (this is not hard to see using lemma 3.7) and we have $[a]_{f,g} \in \mathrm{Hom}(F_f, F_g)$. We have $[1]_{f,g} \in \mathrm{Hom}(F_f, F_g)$ is an isomorphism and its inverse is $[1]_{g,f}$. □

Then the $\mathcal{O}_K$-module structures $(F_f, [.]_f)$ and $(F_g, [.]_g)$ are isomorphic are are thus independent of the choice of $f, g \in \mathcal{P}(\pi)$.

4. Totally ramified abelian extensions

4.1. Building totally ramified abelian extensions.

As we have seen in section 2, adjoining the root of the polynomial $X^{q^n-1} - 1$ to K yields the unique unramified extension $K_n := K(\boldsymbol{\mu}_{q^n-1})$ of degree $n \geq 1$. In this section we will see that we can construct totally ramified extensions of K in somewhat the same manner by considering $K(\boldsymbol{\mu}_{n,f})$ where $\boldsymbol{\mu}_{n,f}$ is a set of roots.

Now that we have introduced formal $\mathcal{O}_K$-modules we finally get to put them to construct the so-called Lubin-Tate extensions. Before we do so however we need to introduce a new player. We keep using the same notations as in section 3. Recall that K^s is the separable closure of K in K^{al}, we define $\boldsymbol{\mu} := \{x \in K^s, |x| < 1\}$ which we endow with the $\mathcal{O}_K$-module structure $(F_f, [.]_f)$. We also define the following set

$$\boldsymbol{\mu}_{n,f} := \{x \in \boldsymbol{\mu}, [\pi^n]_f(x) = 0\} = \{x \in \boldsymbol{\mu}, f^{(n)}(x) := (f \circ \cdots \circ f)(x) = 0\}$$

Recall that $[.]_f$ is a ring homomorphism and notice that $[\pi]_f = f$ so the last equality holds. The set $\boldsymbol{\mu}_{n,f}$ is actually a sub-module of $(\boldsymbol{\mu}, F_f, [.]_f)$ and we have the following proposition

Proposition 4.1. *$\boldsymbol{\mu}_{n,f}$ is isomorphic to $\mathcal{O}_K/(\pi^n)$ as an $\mathcal{O}_K$-module.*

Proof. From proposition 3.10, for any $g \in \mathcal{P}(\pi)$ we have an $\mathcal{O}_K$-module isomorphism from $(F_f, [.]_f)$ to $(F_g, [.]_g)$ so it suffices to prove that $(\boldsymbol{\mu}_{n,g}, F_g, [.]_g) \cong \mathcal{O}_K/(\pi^n)$ for $g(X) := \pi X + X^q \in \mathcal{P}(\pi)$. The power series $g^{(n)}$ is actually a polynomial in $\mathcal{O}_K$ so it has a finite set of roots in so the $\mathcal{O}_K$-module $\boldsymbol{\mu}_{n,g}$ is finite. Then since $\mathcal{O}_K$ is a principal ideal domain (even better it is a DVR) we deduce by the structure theorem

$$\boldsymbol{\mu}_{n,f} \cong \mathcal{O}_K/(\pi^{r_1}) \times \cdots \times \mathcal{O}_K/(\pi^{r_s})$$

where $r_1 \leq \cdots \leq r_s$ are positive integers. The polynomial $X^{q-1} + \pi$ is separable because it is co-prime with its derivative $(q-1)X^{q-1} \neq 0$ (because $\mathrm{char}(K)$ does not divide $q-1$). Then the polynomial $g(X) = X(X^{q-1} + \pi)$ has q-distinct roots in K^s. The non zero roots are conjugates so they have the same valuation and their product is $\pm(\pi)$ thus all the roots of g have positive valuation and thus lie are all in $\boldsymbol{\mu}$. This means that $|\boldsymbol{\mu}_{1,g}| = q$ so since $|\mathcal{O}_K/(\pi^m)| = q^m$ for any $m \geq 1$ we deduce from the structure theorem above that $\boldsymbol{\mu}_{1,f} \cong \mathcal{O}_K/(\pi)$ to deduce the general case we use induction with the following exact sequence

$$0 \to \boldsymbol{\mu}_{1,g} \to \boldsymbol{\mu}_{n,g} \xrightarrow{[\pi]_g} \boldsymbol{\mu}_{n-1,g} \to 0$$

the map $\boldsymbol{\mu}_{1,g} \to \boldsymbol{\mu}_{n,g}$ is clearly injective (it's actually just an inclusion map) so to see that we have an exact sequence we only need the surjectivity of the second map. For any $\alpha \in \boldsymbol{\mu}$ by the same argument as above the roots of $g(X) - \alpha$ have a positive valuation and thus are in $\boldsymbol{\mu}$ which means that $[\pi]_g : \boldsymbol{\mu} \to \boldsymbol{\mu}$ is a surjection which immediately give us the surjectivity of $\boldsymbol{\mu}_{n,g} \xrightarrow{[\pi]_g} \boldsymbol{\mu}_{n-1,g}$ so we have indeed an exact sequence. We then deduce that $\boldsymbol{\mu}_{n,g} \cong \boldsymbol{\mu}_{1,g} \times \boldsymbol{\mu}_{n-1,g}$ which gives by induction $|\boldsymbol{\mu}_{n,g}| = q^n$ also by induction we get that $\boldsymbol{\mu}_{n,g}$ is cyclic. Combining this with the structure theorem and the fact that $\boldsymbol{\mu}_{n,g}$ contains a subgroup isomorphic to $\mathcal{O}_K/(\pi^n)$ □

For any finite Galois extension L/K, $x_1, \ldots, x_n \in \mathfrak{m}_L$ and power series $F \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ the power series $F(x_1, \ldots, x_n)$ converges in L and for $\sigma \in \mathrm{Gal}(L/K)$ we have $\sigma(F(x_1, \ldots, x_n)) = F(\sigma(x_1), \ldots, \sigma(x_n))$ by taking limits in L. We then have a group action of $\mathrm{Gal}(L/K)$ on $\boldsymbol{\mu}_{n,f}$ and the action is compatible with the $\mathcal{O}_K$-module structure of $\boldsymbol{\mu}_{n,f}$. We denote by $K_{\pi,n} := K(\boldsymbol{\mu}_{n,f})$ and we recall that $f \in \mathcal{P}(\pi)$ which means that a priori the extension $K_{\pi,n}$ depends on both π and f. But since as we seen before the $\mathcal{O}_K$-modules $\boldsymbol{\mu}_{n,f}$ are isomorphic to one another when $f \in \mathcal{P}(\pi)$ the extension $K_{\pi,n}$ does not depend on f hence the notation $K_{\pi,n}$. Observe that $K_{\pi,n}/K$ is by definition a Galois extension because it is the splitting field of a polynomial (choose $f(X) = X^q + \pi X$). We begin with the following important theorem.

Theorem 4.2.

i) *For any* $n \geq 1, K_{\pi,n}/K$ *is a totally ramified extension and* $[K_{\pi,n} : K] = (q-1)q^{n-1}$

ii) *The action of* $\mathcal{O}_K$ *on* $\boldsymbol{\mu}_{n,f}$ *induces a group isomorphism* $(\mathcal{O}_K/(\pi)^n)^\times \to \operatorname{Gal}(K_{n,\pi}/K)$

iii) $\pi \in N(K_{\pi,n})$

Proof. Since $K_{\pi,n}$ does not depend on the choice of f we pick $f = X^q + \pi X$. Since the map $[\pi]_f : \boldsymbol{\mu} \to \boldsymbol{\mu}$ is surjective we can inductively define a sequence $x_1, \ldots, x_n$ such that $f(x_1) = 0$ and $f(x_{i+1}) = x_i$ for any $1 \leq i \leq n-1$ and we the extensions $K(x_i)$ and we have

$$K \subset K(x_1) \subset \cdots \subset K(x_n) \subset K_{\pi,n}$$

For each $i \leq n-1$ the extension $K(x_{i+1})/K(x_i)$ is obtained by adding the root x_{i+1} of $P_i(X) =: f(X) - x_i = X^q + \pi X - x_i \in \mathcal{O}_{K(x_i)}[X]$ which is an Eisentein polynomial. The the first extension of previous tower has degree $q-1$ and all the others have degree q and they are all totally ramified extensions which means that $K(x_n)/K$ is totally ramified. Thus we deduce that we necessarily have $[K_{\pi,n} : K] \geq (q-1)q^{n-1}$. It remains to show that $K(x_n) = K_{\pi,n}$.

$K_{\pi,n} := K(\boldsymbol{\mu}_{n,f})$ is the splitting field of $f^{(n)}$ which means that the Galois group $\operatorname{Gal}(K_{\pi,n})$ is isomorphic to a subgroup in the group of permutations of $\boldsymbol{\mu}_{n,f}$ even better the action of $\operatorname{Gal}(K_{\pi,n})$ is compatible with the $\mathcal{O}_K$-module structure of $\boldsymbol{\mu}_{n,f}$ thus $\operatorname{Gal}(K_{\pi,n}/K)$ is actually isomorphic to a subgroup of $Aut_{\mathcal{O}_K}(\boldsymbol{\mu}_{n,f})$ and from proposition 4.1 we have $\boldsymbol{\mu}_{n,f} \cong \mathcal{O}_K/(\pi^n)$ then we have $Aut_{\mathcal{O}_K}(\boldsymbol{\mu}_{n,f}) \cong (\mathcal{O}_K/(\pi^n))^\times$. This means that $|\operatorname{Gal}(K_{\pi,n})| \leq |(\mathcal{O}_K/(\pi^n))^\times| = (q-1)q^{n-1}$. Then we deduce that $K_{\pi,n} = K(x_n)$ and this finishes the proof of (i). The second point (ii) is a consequence of the cardinality bounds we invoked.

For the last point in the theorem, notice that $x_n \neq 0$ is a root of the polynomial $f^{(n)}(X)$ then it is a also a root of $Q(X) = (X^{q-1} + \pi) \circ f^{(n-1)}$ so since $[K[x_n] : K] = (q-1)q^{n-1}$ the polynomial Q has to be both the minimal and characteristic polynomial of x_n over K. This allows us to compute the norm of x_n and we have $N_{K_{\pi,n}/K}(x_n) = \pi$. □

We have just constructed as promised a totally ramified extension $K_{\pi,n}$ that is an abelian Galois extension thanks to the second point in theorem 4.2. We define $K_\pi = \cup_{n \geq 1} K_{\pi,n}$ and we then K_π is a Galois extension and we have $\operatorname{Gal}(K_\pi/K) \cong \varprojlim_n (\mathcal{O}_K/(\pi^n))^\times = \mathcal{O}_K^\times$ which means that K_π is also abelian. The extension K_π depends on the choice of π as we shall see shortely. Nevertheless we have the following nice result

Proposition 4.3. *If* $u \in \mathcal{U}_n := 1 + \mathfrak{m}_K^n = 1 + (\pi^n)$ *and* $\pi' = u\pi$ *where* π *is a uniformizer for* K *then we have* $K_{\pi',n} = K_{\pi,n}$ *for any* n.

Proof. Let $u = 1 + \pi^n v \in 1 + \mathfrak{m}_K^n$ where $v \in \mathcal{O}_K$ and let $\pi' = u\pi$. We first claim that there exists an element $\alpha \in \mathcal{O}_{K^s}$ (where K^s is the separable closure of K) such that $u = \frac{\operatorname{Frob}(\alpha)}{\alpha}$. To see this we show that we can construct α in such a way that $\operatorname{Frob}(\alpha)/\alpha \equiv u \mod \pi^k$ for all $k \geq 0$.

Let's look for α of the form $\alpha = 1 + \pi^n w$. We then have $\frac{\operatorname{Frob}(\alpha)}{\alpha} = \frac{1+\operatorname{Frob}(\pi^n w)}{1+\pi^n w}$ and thus we get $\frac{\operatorname{Frob}(\alpha)}{\alpha} \equiv (1+\operatorname{Frob}(\pi^n w))(1-\pi^n w) \equiv 1+\operatorname{Frob}(\pi^n w) - \pi^n w \mod (\pi^{n+1})$. Then for a start we need to have $1+\operatorname{Frob}(\pi^n w) - \pi^n w \equiv 1 + \pi^n v \mod (\pi^{n+1})$. Since Galois action preserves valuations we can write $\operatorname{Frob}(\pi^n) = \pi^n a$ thus we need to have $a \operatorname{Frob}(w) - w \equiv v \mod (\pi)$ taking the quotient modulo (π) we get need to solve

$\overline{a} \times \overline{w}^q - \overline{w} = \overline{v}$ in the residue field. Then we can find $\overline{w}$ satisfying this equation. Following this procedure we can produce such an element α.

Now let $f \in \mathcal{P}(\pi), g \in \mathcal{P}(\pi')$ and $\beta_g \in \boldsymbol{\mu}_{n,g}$, using similar arguments to the proof of lemma 3.7 (see [Yos08, Lemma 3.4]) we can prove that there exists a unique power series Φ such that $\Phi(X) \equiv \alpha X \mod (\deg 2)$ and Φ satisfies the commutation relation $g \circ \Phi = \Phi^{\mathrm{Frob}} \circ f$ where Φ^{Frob} is the power series whose coefficients are obteined by applying Frob to those of Φ. Then we have $\Phi \in \mathrm{Hom}(F_f, F_g)$ and also $\Phi \circ F_f \equiv \alpha(X+Y) \mod (\deg 2) \equiv F_g \circ \Phi$. Since $F_f, F_g \in \mathcal{O}_K[[X,Y]]$ we have $F_f^{\mathrm{Frob}} = F_f$ and thus we deduce that $g \circ \Phi \circ F_f = \Phi^{\mathrm{Frob}} \circ f \circ F_f = \Phi^{\mathrm{Frob}} \circ F_f \circ f = \Phi^{\mathrm{Frob}} \circ F_f^{\mathrm{Frob}} \circ f$. This means that $g \circ \Phi \circ F_f = (\Phi \circ F_f)^{\mathrm{Frob}} \circ f$. In the same manner we also get $g \circ F_g \circ \Phi = (F_g \circ \Phi)^{\mathrm{Frob}} \circ f$ and then by uniqueness of Φ we deduce that $\Phi \circ F_f = F_g \circ \Phi$. From this we get $\boldsymbol{\mu}_{n,g} = \Phi(\boldsymbol{\mu}_{n,f})$. Since $\beta_g \in \boldsymbol{\mu}_{n,g}$ there exists $\beta_f \in \boldsymbol{\mu}_{n,f}$ such that $\beta_g = \Phi(\beta_f)$.

We have $g \equiv \pi' X \equiv u\pi X \mod (\deg 2)$ then we get $\Phi \circ [\pi']_f = [\pi']_g \circ \Phi$. This equation translates to $\Phi \circ [u]_f \circ [\pi]_f = g \circ \Phi = \Phi^{\mathrm{Frob}} \circ f = \Phi^{\mathrm{Frob}} \circ [\pi]_f$. We then get $\Phi^{\mathrm{Frob}} = \Phi \circ [u]_f$. Applying this to $\beta_f \in \boldsymbol{\mu}_{n,f}$ yields $\Phi_{\mathrm{Frob}}(\beta_f) = \Phi([u]_f(\alpha))$. But since $u \in 1 + \mathfrak{m}_K^n$ and we have seen that $Aut_{\mathcal{O}_K}(\boldsymbol{\mu}_{n,f}) \cong (\mathcal{O}_K/(\pi^n))^\times = \mathcal{O}_K^\times/(1+\mathfrak{m}_K^n)$ the map $[u]_f$ is the identity map in $Aut(\boldsymbol{\mu}_{n,f})$ thus we deduce that $\Phi^{\mathrm{Frob}}(\beta_f) = \Phi(\beta_f)$. Galois theory gives $K^{ur} \cap K_{n,\pi} = K$, the the Frobenius map Frob can be extended to an automorphism of the field $L_n := K^{ur}.K_{\pi,n}$ such that Frob acts trivially on $K_{n,\pi}$. Now since $\mathrm{Frob}(\beta_f) = \beta_f$ we get $\Phi^{\mathrm{Frob}}(\beta_f) = \mathrm{Frob}(\Phi(\beta_f)) = \Phi(\beta_f)$. Thus we deduce that $\beta_g = \Phi(\beta_f)$ is fixed by Frob which means that $\beta_g \in K_{n,\pi}$ this being valid for any $\beta_g \in \boldsymbol{\mu}_{n,g}$ we get $K_{\pi',n} \subset K_{\pi,n}$. A similar proof with u^{-1} gives the other inclusion. □

The proposition states that to have $K_{\pi,n} = K_{\pi',n}$ for two uniformizers π, π' it suffices to have $\pi'/\pi \equiv 1 \mod \mathfrak{m}_K^n$. In some sense as n becomes bigger it get harder and harder to have $K_{\pi,n} = K_{\pi',n}$ when π, π' are fixed. We now proceed to study norm groups which are going to give a deeper understanding for the role of the uniformizer π.

4.2. **Norm group.**

We denote by N the the norm map of the extension $K_{\pi,n}/K$. We have $N(K_{\pi,n}^\times)$ is a subgroup of $K^\times$.

Proposition 4.4. *$N_{\pi,n}(K_{\pi,n}^\times)$ as a subgroup of $K^\times$ is generated by π and $1 + \mathfrak{m}_K^n$.*

Proof. Assume the existence of the Artin map of Theorem 1.1 (which we have not discussed yet). Then by choosing $L = K_{\pi,n}$ which is a finite abelian extension, point (ii) of the theorem ensures that $K^\times/N(L^\times) \cong \mathrm{Gal}(L/K)$. By theorem 4.2 we have $\mathrm{Gal}(L/K) \cong \mathcal{O}_K^\times/(1+\mathfrak{m}_K^n)$ then we deduce that $K^\times/N(L^\times) \cong \mathcal{O}_K^\times/(1+\mathfrak{m}_K^n) = K^\times/\langle \pi, 1+\mathfrak{m}_K^n\rangle$. Thus we deduce that $N^\times = \langle \pi, 1+\mathfrak{m}_K^n\rangle$ as stated.

Without assuming the existence of the Artin map the proof is considerably more complicated. The interested reader can refer to [Rie06]. □

We then have $N_{K_\pi/K}(K_\pi^\times) = \cap N_{\pi,n}(K_{\pi,n}^\times) = \pi^{\mathbb{Z}}$. This shows that the field extensions K_π are distinct for different choices of the uniformizer π which confirms that there is no canonical maximal totally ramified extension. In the next section however we will show that $K^{ab} = K^{ur}.K_\pi$ for any uniformizer $\pi \in K$ and that we have canonical totally ramified extensions of K^{ur}.

5. Artin's map

The motivation behind building totally ramified abelian extension in the last section is to prepare for the proof of Theorem 1.1 which we finally get to discuss in this section. Let $L_\pi = K_\pi.K^{ur}$ which an extension of K that still depends on π a priori. Since $K_\pi \cap K^{ur}$ (thanks to the nature of their respective Galois groups $\mathcal{O}_K^\times$ and $\widehat{\mathbb{Z}}$) we have a group isomorphism

$$\mathrm{Gal}(L_\pi/K) \xrightarrow{\sim} \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{ur}/K) : \sigma \mapsto (\sigma_{|K_\pi}, \sigma_{|K^{ur}})$$

It suffices then to have the action of σ on the two sub-extensions of K_π and K^{ur} to describe σ on L_π. We can then define a homomorphism $\phi_\pi : K^\times \to \mathrm{Gal}(L_\pi/K)$ via the map

$$\begin{aligned} K^\times &\longrightarrow \mathcal{O}^\times \times \mathbb{Z} \longrightarrow \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{ur}/K) \cong \mathrm{Gal}(L_\pi/K) \\ u\pi^r &\mapsto \qquad (u, r) \mapsto ([u^{-1}]_f, \mathrm{Frob}^m) := \phi_\pi(u\pi^r) \end{aligned}$$

The moral of this story is that we are hoping to have $L_\pi = K^{ab}$ for any uniformizer π and then our goal would be to prove that ϕ_π (which is then independent of π is the desired Artin map. Let's start by proving that L_π and ϕ_π are actually independent of π.

Let $\pi' = u\pi$ another uniformizer of K where $u \in \mathcal{O}_K^\times$ and let $f \in \mathcal{P}(\pi), g \in \mathcal{P}(\pi')$ (we are using notation from section 3) and F_f, F_g their corresponding Lubin-Tate formal groups. If we have $F_f \cong F_g$ over $\mathcal{O}_K$ then the two extensions $K_{\pi,n}, K_{\pi',n}$ would be equal for any $n \geq 1$. Since $\mathcal{O}_K^\times \ni u = \pi'/\pi \neq 1$ there exists $n \geq 1$ such that $u \notin 1 + \mathfrak{m}_K^n$. Then using proposition 4.4 we see that $\pi \in N(K_{\pi,n}^\times)$ but we have $\pi' \notin N(K_\pi^\times) = (\pi, 1 + \mathfrak{m}_K^n)$. This gives us the converse of proposition 4.3 and we then have $K_{\pi,r} = K_{\pi',r} \iff \pi'/\pi \in 1 + \mathfrak{m}_K^r$ for any integer $r \geq 1$. We want to show that the extensions $K_{\pi,n}.K^{ur}$ are the canonical totally ramified extensions of K^{ur} for this we need to prove that F_f, F_g are isomorphic over K^{ur} i.e there exists $\Phi \in X\mathcal{O}_{K^{ur}}[[X]]$ such that Φ is invertible and $\Phi : F_f \to F_g$ is a homomorphism.

Every infinite algebraic extension of K is not complete in particular K^{ur} is not complete which means that we may have problems when evaluating power series on $\mathfrak{m}_{K^{ur}}$. It is then handy to work on the completion of K^{ur} which we denote as usual by $\widehat{K^{ur}}$. We can uniquely extend the automorphism Frob of K^{ur} to $\widehat{K^{ur}}$ by continuity and we keep the notation Frob for the extended automorphism. For our future arguments in this section we will need the following

Lemma 5.1. *There exists a power series $\Phi \in \widehat{\mathcal{O}_{K^{ur}}}[[X]]$ satisfying the following*

$$\begin{aligned} &i) \quad \Phi(X) \equiv uX \mod (\deg 2) \\ &ii) \quad \mathrm{Frob} \circ \Phi = \Phi \circ [u]_f \\ &iii) \quad \Phi \circ F_f = F_g \circ \Phi \\ &iv) \quad \Phi \circ [a]_f = [a]_g \circ \Phi, \quad \forall\, a \in \mathcal{O}_K \end{aligned}$$

Properties (i) translates to Φ is invertible and (iii) translates to $\Phi \in \mathrm{Hom}(F_f, F_g)$ then together they give us that Φ is an isomorphism from F_f to F_g. The point (iv)

means that Φ commutes with the $\mathcal{O}_K$-module actions $[.]_f, [.]_g$. This result is proven in [Mil, Proposition 3.10, p 27] and we will not go through the proof in this paper.

Since $F_f \cong F_g$ over $\widehat{K^{ur}}$ the sets $\boldsymbol{\mu_{n,f}}, \boldsymbol{\mu_{n,g}}$ are isomorphic as $\widehat{\mathcal{O}_{K^{ur}}}$-modules and thus the extensions $K_\pi.\widehat{K^{ur}} = K_{\pi'}.\widehat{K^{ur}}$ are the same. Taking completions are also the same and we then get $\widehat{K_{\pi'}.K^{ur}} = \widehat{K_\pi.K^{ur}}$ which means $\widehat{L_\pi} = \widehat{L_{\pi'}}$. The following lemma allows us to conclude that $L_\pi = L_{\pi'}$

Lemma 5.2. *Let L a separable algebraic extension of a local field K and $\widehat{L}$ its completion. Then we have $\widehat{L} \cap K^s = L$.*

Proof. $\mathrm{Gal}(K^s/L)$ acts trivially on L thus by continuity its also trivial on $\widehat{L} \cap K^s$. We then deduce that $\widehat{L} \cap K^s \subset L$. The other inclusion is trivial. □

Proof. (ϕ_π does not depend on π)
Now we know that L_π does not depend on π we denote it L_K and we now need to show that also ϕ_π does not depend on π. We start by proving $\phi_{\pi'}(\pi') = \phi_{\pi'}(\pi)$ for any couple of uniformizers $\pi, \pi' := u\pi$ where $u \in \mathcal{O}_K^\times$.

By definition the automorphisms $\phi_{\pi'}(\pi'), \phi_{\pi'}(\pi) \in \mathrm{Gal}(L_K/K)$ satisfy $\phi_{\pi'}(\pi')_{|K^{ur}} = \phi_{\pi'}(\pi)_{|K^{ur}} = \mathrm{Frob}$, it remains to show that the they are also the same on $K_{\pi'}$. Notice that $\phi_{\pi'}(\pi')$ is the identity map on $K_{\pi'}$ so we need only show that $\phi_{\pi'}(\pi)$ is also the identity map on $K_{\pi'}$. Let Φ as in Lemma 5.1 which is an isomorphism $\Phi : F_f \xrightarrow{\sim} F_g$. Recall that Φ is also an isomorphism from $\boldsymbol{\mu_{n,f}}$ to $\boldsymbol{\mu_{n,g}}$ that's compatible with the $\mathcal{O}_K$-module structures. Since $K_{\pi',n} = K(\boldsymbol{\mu_{n,g}})$ it suffices to show that $\phi_{\pi'}(\pi)$ fixes all the elements in $\boldsymbol{\mu_{n,g}}$ for all n.

Let $\beta_g \in \boldsymbol{\mu_{n,f}}$, there exits $\beta_f \in \boldsymbol{\mu_{n,f}}$ such that $\beta_g = \Phi(\beta_f)$. We have

$$\begin{aligned}
\phi_{\pi'}(\pi)_{|K_\pi}(\beta_g) = (\phi_{\pi'}(u^{-1})\phi_{\pi'}(\pi'))_{|K_\pi}(\beta_f) &= [u]_g(\beta_g) \\
&= [u]_g \circ \Phi(\beta_f) \\
&= \Phi \circ [u]_f(\beta_f) \\
&= \Phi \circ \phi_\pi(\pi)(\beta_f) \\
&= \Phi(\beta_f) = \beta_g
\end{aligned}$$

then the automorphisms $\phi_{\pi'}(\pi'), \phi_{\pi'}(\pi)$ are the same on both K_π and K^{ur} so they are the same on $L_K = K_\pi.K^{ur}$ i.e $\phi_{\pi'}(\pi') = \phi_{\pi'}(\pi)$. To see that $\phi_\pi = \phi_{\pi'}$ consider three uniformizers $\pi, \pi', \pi'' \in K$ we have $\phi_{\pi'}(\pi'') = \phi_{\pi''}(\pi'') = \phi_\pi(\pi'')$ so we have $\phi_{\pi'}(\pi'') = \phi_\pi(\pi'')$ for any uniformizer π'' and since the set of uniformizers generate $K^\times$ we deduce that $\phi_\pi = \phi_{\pi'}$ we then denote this map ϕ_K. □

To finish the proof of Theorem 1.1 we have to prove that $L_K = K^{ab}$ and ϕ_K satisfies the second property (ii).

5.1. Partial proof of $K^{ab} = L_K$.

The Frobenius map Frob is defined on K^{ur} which is a sub-extension of K^{ab}, then using Steiniz's theorem there exists a lift $\varphi \in Gal(K^{ab}/K)$ such that $\varphi_{|K^{ur}} = \mathrm{Frob}$. Let L^φ the field of elements in K^{ab} fixed by φ. Obviously, since $\mathrm{Gal}(K^{ur}/K)$ is generated by Frob we have $K^{ur} \cap L^\varphi = K$. We have the following preliminary result.

Lemma 5.3. $L^\varphi.K^{ur} = K^{ab}$.

Proof. Let $L^\varphi \subset L \subset K^{ab}$ an intermediate extension such that $[L : L^\varphi] = n$ is finite. We want to show that $L = L^\varphi.K_n$. Now let $L' = L^\varphi.K_n$ and $L'' = L.K_n$ recall that K_n is the unique unramified extension of K of degree n. Since $L^\varphi \cap K^{ur} = K$ we have $[L' : L^\varphi] = [K_n : K] = n$. Then L'' is a finite extension of L^φ and it contains two sub-extensions of L^φ of degree n which are L and L'. Since L^φ is by definition the field of elements fixed by φ, the group $\mathrm{Gal}(L''/F^\varphi)$ must be cyclic generated by $\varphi_{|L''}$ (otherwise φ would fix a larger field than L^φ). Since the extension L'' contains two extensions L and L' of the same degree, their Galois groups of of the same order are are subgroups of the same cyclic group, thus they are equal. We then deduce that $L = K_n L^\varphi$. This being true for any finite extension L/L^φ we deduce that $L^\varphi.K^{ur} = K^{ab}$ □

Thanks to Lemma 5.3, to show that $K^{ab} = K_\pi.K^{ur}$ it suffices to show that $K_\pi = L^\varphi$ for some choice of extension φ of the Frobenius automorphism to K^{ab}.

By definition the map ϕ_K (which we have denoted ϕ_π until proven independent of π) satisfies $\phi_K(x)_{|K^{ur}} = \mathrm{Frob}^m$ for $x \in K^\times$ and $m := v_K(x)$. Then its image $\phi_K(K^\times)$ consists of elements σ in $\mathrm{Gal}(L_K/K)$ such that $\sigma_{|K^{ur}}$ is a power of the Frobenius map. This is actually a dense set since the Frobenius map generates $\mathrm{Gal}(K^{ur}/K) = \widehat{\mathbb{Z}}$. We denote $\sigma = \varphi_{|L_K}$. Since $K^{ur} \subset L_K$ the automorphism σ of L_K is a lifting of Frob to L_K and thus by definition of the map ϕ_K there exists π a uniformizer such that $\sigma = \phi_K(\pi)$ which depends on the original lifting φ. Since the element $\sigma = \varphi_{|L_K}$ fixes the elements of K_π (this can be seen from the definition of ϕ_K and the fact that $K^{ur} \cap K_\pi = K$) we have $K_\pi \subset L^\varphi$. Suppose that L^φ is a strictly bigger field than K_π, then there exits a finite cyclic extension E/K such that $E \subset L^\varphi$ but $E \not\subset K_\pi$. We then have $E \cap K^{ur} = K$ and thus E is totally ramified. Now we define the ramification groups $G_i(E/K) = \{\sigma \in \mathrm{Gal}(E/K), v_E(x - \sigma(x) \geq i+1) \forall\ x \in E\}$ for any $i \geq -1$. Since E is totally ramified we know that $\mathrm{Gal}(E/K) = G_0$ and that $k_E = k = \mathbb{F}_q$. Using the higher ramification group argument we know that $[E : K]$ is the product of $q-1$ and a power of p. We have $[E : E \cap K_{\pi,1}] = [E.K_{\pi,1} : K_{\pi,1}]$ is a power of p and $[E \cap K_{\pi,1} : K]$ is prime to p. We then deduce the existence of a cyclic extension E' of K of p-power degree such that $E' \cap (E \cap K_{\pi,1}) = K$ and that also satisfies $E'.(E \cap K_{\pi,1}) = E$. Clearly $E' \not\subset K_\pi$ because $E \not\subset K_\pi$. It suffices to replace E' if necessary by a subfield and we then get a cyclic p-power extension E' such that $K \subset E' \subset L^\varphi$ and $[E' : E' \cap K_{\pi,1}] = p$.

Now to conclude the proof, we use the field E' to build a totally ramified cyclic extension F/K that has degree p and satisfies $N(F^\times) = K^\times$ which will contradict this lemma whose proof is omitted (refer to [Rie06] for details)

Lemma 5.4. *For any cyclic extension L/K of degree p we have $N(F^\times) \neq K^\times$.*

All that remains is to build such an extension to conclude by contradiction that $L^\varphi = K_\pi$. We shall not go further in the proof and we encourage the interested reader to refer to [Rie06] for a detailed construction of F. We recall however our main goal from all this discussion which was to prove that

$$L^\varphi = K_\pi$$

Armed with this information we see using Lemma 5.3 that $K^{ab} = K_\pi.K^{ur} = L_K$.

Proof property (ii) **in Theorem 1.1 .**

At this stage we have a map $\phi_K : K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$ such that condition (i) of Theorem 1.1 is satisfied before we prove that ϕ_K satisfies the second property (ii) we first ask answer an interesting question which will help us on the long run.

Let L/K be a finite extension. Then L is also a local field and we then have a map $\phi_L : L^\times \longrightarrow \mathrm{Gal}(L^{ab}/L)$ and the question is how does ϕ_L relate to ϕ_K? The following theorem provides an answer.

Theorem 5.5. *Let L/K a finite extension and ϕ_K, ϕ_L the maps built previously the following diagram commutes*

$$\begin{array}{ccc} L^\times & \xrightarrow{\phi_L} & \mathrm{Gal}(L^{ab}/L) \\ \big\downarrow{\scriptstyle N_{L/K}} & & \big\downarrow{\scriptstyle \psi} \\ K^\times & \xrightarrow{\phi_K} & \mathrm{Gal}(K^{ab}/K) \end{array}$$

where ψ is the restriction map i.e $\phi_K(N_{L/K}(x)) = \phi_L(x)_{|K^{ab}}$.

Proof. Omitted. Refer to [Rie06, Theorem 6.7] for a detailed proof. □

Using the theorem above it is no longer too difficult to show property (ii) in Theorem 1.1 and for that we fix a finite abelian extension L/K. We define the following map

$$\phi_{L/K} : K^\times \longrightarrow \mathrm{Gal}(L/K), x \mapsto \phi_K(x)_{|L}$$

and we claim that the map $\phi_{L/K}$ induces an isomorphism $K^\times/N(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K)$. In fact from Theorem 5.5 we can deduce that $\mathrm{Ker}(\phi_{L/K}) = N(L^\times)$. Since the image of ϕ_K contains automorphisms σ such that $\sigma_{|K^{ur}}$ is some power of the Frobenius map it's dense in the group $\mathrm{Gal}(K^{ur}/K)$. Since $\mathrm{Gal}(K^{ab}/L)$ is an open normal subgroup of $\mathrm{Gal}(K^{ab}/K)$ the map $\phi_{L/K}$ is surjective. This means that we can factorize $\phi_{L/K}$ to an isomorphism $K^\times/Ker(\phi_{L/K}) \xrightarrow{\sim} \mathrm{Gal}(L/K)$. We then conclude that he maps ϕ_K induces an isomorphism $K^\times/N(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K)$.

All that remains to show in Theorem 1.1 is the uniqueness of the map ϕ_K which we will then denote Art_K for Artin's map. To see uniqueness let ϕ be another map satisfying the same conditions as ϕ_K. We have $\pi \in N(K_{\pi,n}^\times)$ for any $n \geq 1$. Then using the property (ii) we get $\phi(\pi)_{|K_{\pi,n}}$ and from this we deduce that $\phi_{|K_\pi}$ is the identity map. The first property (i) implies $\phi_K(\pi)_{|K^{ur}} = \phi(\pi)_{|K^{ur}}$. Then we deduce that $\phi(\pi) = \phi_K(\pi)$ for any uniformizer. This means that $\phi = \phi_K$ and hence the maps ϕ_K is unique.

6. Summary and discussion

In this section we take a moment to summaries the steps we went through and to think about the implications of the results that we have seen. First we knew that there exits a maximal unramified extensions of K and in order to get a complete description of K^{ab} we needed to build totally ramified extensions. We then needed Lubin-Tate's formal groups to figure out which elements we need to adjoin to K to produce unramified extensions and we had an $\mathcal{O}_K$-module structure on the sets $\boldsymbol{\mu_{n,f}}$ which allowed us to prove Theorm 4.2. From there we defined maximal totally ramified extensions K_π which turned out not to be canonical but we had however

$K^{ab} = K^{ur}.K_\pi$ and $K^{ur} \cap K_\pi = K$. This allowed us to define the map ϕ_K which we finally proved is the desired Artin map.

From theorem 1.1 we the following commutative diagram for any finite abelian extension L/K

$$\begin{array}{ccc} K^\times & \xrightarrow{\phi} & \mathrm{Gal}(K^{ab}/K) \\ \downarrow & & \downarrow{\scriptstyle \sigma \mapsto \sigma_{|L}} \\ K^\times/N(L^\times) & \xrightarrow{\sim} & \mathrm{Gal}(L/K) \end{array}$$

and we have an isomorphism $K^\times/N(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K)$. Also, for any prime $\pi \in K$, we have $\phi(\pi)_{|K^{ur}}$ is the Frobenius map.

One of the main application of the local Artin reciprocity is classifying Galois extensions of local fields with a fixed abelian Galois group. We refer the reader to [Mil] and [Ser13] for further reading.

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