BURKHOLDER-DAVIS-GUNDY INEQUALITIES

YASSINE EL MAAZOUZ

Abstract. The aim of these notes is to present the argument establishing the Burkholder-Davis-Gundy inequalities

Contents

1. Lévy's characterization of Brownian motion

In this section we tend to some unfinished business from the last sessions: Lévy's charachterization of Brownian motion. At this points this should be a very fast and clear argument thanks to the machinery we have developped (Ito's formula).

Theorem 1.1. *(Lévy) X be a continuous $(\mathcal{F}_t)$-adapted d-dimensional process with $X_0 = 0$. The following are equivalent*

(1) *X is an $(\mathcal{F}_t)$-Brownian motion.*

(2) *X is a continuous local martingale and $\langle X^i, X^j\rangle_t = \delta_{i,j}t$.*

(3) *X is a continuous local martingale and for every choice of functions $f_1, \dots, f_d \in L^2(\mathbb{R}_+)$ the process*

$$\mathcal{E}_t = \exp\left(i\sum_{k=1}^{d}\int_0^t f_k(s)dX_s^k + \frac{1}{2}\sum_{k=1}^{d}\int_0^t f_k^2(s)ds\right)$$

is a complex martingale.

Proof. The fact that (1) $\implies$ (2) is clear. For (2) $\implies$ (3) we use the exponential martingale with the complex coefficient $\lambda = i$ on the local martingale $M_t = \sum_{k=1}^d \int_0^t f_k(s)dX_s^k$ we get that $\mathcal{E}_t = \exp(iM_t - \frac{i^2}{2}\langle M, M\rangle_t)$ is a local martingale. But since $\mathcal{E}$ is bounded it is a complex martingale.

Now assume that (3) holds, then by chooseing $f_k = \xi 1_[0, T]$ for a certain $\xi \in \mathbb{R}^d$ and $T > 0$ we get

Date: June 11, 2020.

$$\mathcal{E}_t = \exp\left(i\langle \xi, X_{t\wedge T}\rangle + \frac{1}{2}|\xi|^2 t\wedge T)\right)$$

is a maringale. Taking $s < t < T$ and using the martingale property we deduce that $X_t - X_s$ is independent of $\mathcal{F}_s$ and has Fourier transform $\mathbb{E}(\exp(i\langle \xi, X_t - X_s\rangle) = \exp(-|\xi|^2(t-s)/2)$. Hence X is indeed a Brownian motion. □

Now that we got that taken care of we will now discuss the Burkholder-Davis-Gundy inequalities (BGD).

2. Burkholder-Davis-Gundy inequalities

In a previous talk we have seen (in the Hilbert space formalism for stochastic integration of continuous semi-martingales) that the normes $\|\|_1$ and $\|\|_2$ defined on the space of L^2-bounded continuous martingales M vanishing at 0 by

$$\|M\|_1 = \mathbb{E}[M_\infty^2]^{1/2} = \mathbb{E}[\langle M, M\rangle_\infty]^{1/2} \text{ and } \|M\|_2 = \mathbb{E}[(M_\infty^*)^2]^{1/2}$$

are equivalent by Doob's inequality. Only first one defines a Hilbert space structure and we have $\|.\|_1 \le \|.\|_2 \le 2\|.\|_1$. It turns out that this fact is a special case of what's called the BDG inequalities which will occupy us in this section.

2.1. Statement and consequences.

Theorem 2.1. *For any $p > 0$ there exist two constants c_p and C_p such that for any continuous local martingale M vanishing at 0 we have*

$$c_p\mathbb{E}[\langle M, M\rangle_\infty^{p/2}] \le \mathbb{E}[(M_\infty^*)^p] \le C_p\mathbb{E}[\langle M, M\rangle_\infty^{p/2}]$$

Let's call $\mathcal{H}^p$ the space of continuous local martingales vanishing at 0 such that M_∞^* is in L^p. The above theorem gives an equivalence of norms on this space. The elements of $\mathcal{H}^p$ are true martingales for $p \ge 1$ and for $p > 1$ they are bounded in L^p. The latter fact is however not true for $p = 1$ because the space $\mathcal{H}^1$ i actually smaller than the space of continuous L^1-bounded martingales (to be checked in Revuz and Yor exercise 3.15).

By stopping at a time stopping time T, Theorem 2.1 yileds the following result which is simple to understand yet very important in applications:

Corollary 2.2. *For a stopping time T one has*

$$c_p\mathbb{E}[\langle M, M\rangle_T^{p/2}] \le \mathbb{E}[(M_T^*)^p] \le C_p\mathbb{E}[\langle M, M\rangle_T^{p/2}]$$

In general for bounded predictable process H we have

$$c_p\mathbb{E}\left[\left(\int_0^T H_s^2 d\langle M, M\rangle_s\right)^{p/2}\right] \le \mathbb{E}\left[\sup_{t\le T}\left|\int_0^T H_s dM_s\right|^p\right] \le C_p\mathbb{E}\left[\left(\int_0^T H_s^2 d\langle M, M\rangle_s\right)^{p/2}\right]$$

One can even work with the integral of a predictable process against M and get a more general inequality. We refer to [RY13] for more details.

The proof of Theorem 2.1 will be broken down to several steps and for that we follow the argument presented in [RY13] which we will expand when needed. The first step is to show the right-hand-side inequlity for $p \geq 2$ and the left-hand-side inequality for $p \geq 4$. We will then show that these two result suffice by reducing the theorem with a domination technique which is the purpose of the next subsection.

2.2. **A proof.**

In the arguments presented here we will write a_p for a constant that depends only on p, which we might change from line to line, since we are only interested in bounding quantities.

Proposition 2.3. *For $p \geq 2$ there exists a constant C_p such that* $\mathbb{E}[(M^*_\infty)^p] \leq C_p\mathbb{E}[\langle M, M\rangle_\infty^{p/2}]$

As we have seen in previous talk, by stopping we can reduce to the case where M is a bounded martingale that vanishes at 0. The proof is very clean using Itô's formula.

Proof. The map $f : x \mapsto |x|^p$ is twice differentiable with $f'(x) = \text{sgn}(x)p|x|^{p-1}$ and $f''(x) = p(p-1)|x|^{p-2}$. Applying Itô's formula we get:

$$M^p_\infty = \int_0^\infty \text{sgn}(M_s)p|M_s|^{p-1}dM_s + \frac{1}{2}\int_0^\infty p(p-1)|M_s|^{p-2}d\langle M, M\rangle_s$$

Taking the expectation of the this equation we get

$$\begin{aligned}\mathbb{E}[|M_\infty|^p] &= \frac{p(p-1)}{2}\mathbb{E}\left[\int_0^\infty |M_s|^{p-2}d\langle M, M\rangle_s\right] \\ &\leq \frac{p(p-1)}{2}\mathbb{E}\left[|M^*_\infty|^{p-2}\langle M, M\rangle_\infty\right]\end{aligned}$$

Hölder's inequality with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ give us the following

$$\mathbb{E}\left[|M^*_\infty|^{p-2}\langle M, M\rangle_\infty\right] \leq \mathbb{E}\left[|M^*_\infty|^p\right]^{(p-2)/p}\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right]^{p/2}$$

Now we deduce that

$$\mathbb{E}[|M_\infty|^p] \leq \mathbb{E}\left[|M^*_\infty|^p\right]^{(p-2)/p}\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right]^{p/2}$$

Doob's maximal inequality gives us $E[|M^*_\infty|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_\infty|^p\right]$ so that combining this with the last inequality would give

$$\mathbb{E}[|M^*_\infty|^p] \leq C_p\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right]$$

□

Notice that the condition $p \geq 2$ is crucial since we need differentiability to apply Itô 's formula. The next result is the left-hand-side inequality for $p \geq 4$.

Proposition 2.4. *For $p \geq 4$ there exists a constant C_p such that* $\mathbb{E}[(M^*_\infty)^p] \leq C_p\mathbb{E}[\langle M, M\rangle_\infty^{p/2}]$

Again we reduce to the case where M is bounded by stopping. The proof uses Itô's formula once more.

Proof. The convexity of $x \mapsto |x|^p$ gives $|x+y|^p \leq a_p(|x|^p + |y|^p)$ for a certain constant a_p. Now comes the time to deploy Itô's formula and we write

$$M_t^2 = 2\int_0^t M_s dM_s + \langle M, M\rangle_t$$

By rearanging terms and using the converxity (and sending t to ∞ since M is bounded) inequality above we deduce

$$\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right] \leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \mathbb{E}\left[\left| \int_0^\infty M_s dM_s \right|^{p/2} \right] \right)$$

Proposition 2.3 applies to the local martingale $\int_0^t M_s dM_s$ so we get the following

$$\begin{aligned}\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right] &\leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \mathbb{E}\left[\left| \int_0^\infty M_s^2 d\langle M, M\rangle_s \right|^{p/4} \right] \right) \\ &\leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \mathbb{E}\left[(M_\infty^*)^{p/2} \langle M, M\rangle_\infty^{p/4} \right] \right) \\ &\leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \left(\mathbb{E}\left[(M_\infty^*)^p \right] \mathbb{E}\left[\langle M, M\rangle_\infty^{p/2} \right] \right)^{1/2} \right)\end{aligned}$$

Set $A = \left(\mathbb{E}\left[\langle M, M\rangle_\infty^{p/2} \right] \right)^{1/2}$ and $B = (\mathbb{E}[(M_\infty^*)^p])^{1/2}$ so that that above inequality can be rewritten as

$$A^2 - a_p AB - a_p B^2 \leq 0$$

This means that A is less than the positive root of the polynomial $X^2 - a_p yX - a_p y^2$ which is of the form $a_p y$ (the constant a_p may have changed!!). So that $A \leq a_p B$ which proves the desired result. $\square$

Notice that we needed $p \geq 4$ becasue we applied Proposition 2.3 with $p/2$. Now we see that the inequalities hold for $p \geq 4$ and that they are at least plausible for every p. The next step is to reduce Theorem 2.1 so that these two results suffice. This reduction is done by a domination technique which we will now explain.

Definition 2.5. A positive adapted right-continuous process X is dominated by an increasing process A if for any bounded stopping time T one has

$$\mathbb{E}[X_T|\mathcal{F}_0] \leq \mathbb{E}[A_T|\mathcal{F}_0]$$

First we start with the following useful lemma

Lemma 2.6. *If X is dominated by A and A is continuous then for $x, y > 0$ we have*

$$\mathbb{P}(X_\infty^* > x, A_\infty \leq y) \leq \frac{1}{x}\mathbb{E}[A_\infty \wedge y]$$

where $X_\infty^* = \sup_s X_s$

Proof. It suffices to prove the inequality in the case $\mathbb{P}(A_0 \leq y) > 0$ (otherwise it obviously holds since A is increasing) and actually we can throw away the event $(A_0 > y)$ by conditionning on its complement so we reduce to the case where $\mathbb{P}(A_0 \leq y) = 1$ (this is by replacing by the conditional probability under which the domination hypothesis still holds).

We reduce the problem further using Fatou's lemma to the following: it is enough to show that

$$\mathbb{P}(X_n^* > x, A_n \leq y) \leq \frac{1}{x} E[A_\infty \wedge y]$$

Notice that working on $[0, n]$ is the same as working on $[0, \infty]$ and assuming that X_∞ exists and that the domination is true for any stopping time bounded or not (by a simple time change).

Now we define $R = \inf\{t : A_t > y\}$ and $S = \inf\{t : X_t > x\}$ where in both definition the infimum of the empty set is $+\infty$. Since A is increasing we have $\{A_\infty \leq y\} = \{R = \infty\}$ so that

$$\begin{aligned}
\mathbb{P}(X_\infty^* > x; A_\infty \leq y) &= \mathbb{P}(X_\infty^* > x; R = \infty) \\
&\leq \mathbb{P}(X_S \geq x; (S < \infty) \cap (R = \infty)) \\
&\leq \mathbb{P}(X_{S\wedge R} \geq x) \\
&\leq \frac{1}{x}\mathbb{E}[X_{S\wedge R}] \leq \frac{1}{x}\mathbb{E}[A_{S\wedge R}] \leq \frac{1}{x}\mathbb{E}[A_\infty \wedge y]
\end{aligned}$$

□

Now we present the final result that will allow us to reduce theorem 2.1 to the two results already shown.

Proposition 2.7. *Unde the conditions of Lemma 2.6 for any $0 < k < 1$ we have*

$$\mathbb{E}[(X_\infty^*)^k] \leq \frac{2-k}{1-k}\mathbb{E}[A_\infty^k]$$

The proof is not very hard and we can already see how this can help us finish the proof of theorem 2.1.

Proof. Let F be a continuous increasing function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with $F(0) = 0$. Fubini's theorem combined with Lemma 2.6 give us

$$\begin{aligned}
\mathbb{E}[F(X_\infty^*)] &= \mathbb{E}\left[\int_0^\infty 1_{X_\infty^* > x} dF(x)\right] \\
&\leq \int_0^\infty \left(\mathbb{P}(X_\infty^* > x, A_\infty \leq x) + \mathbb{P}(A_\infty > x)\right) dF(x) \\
&\leq \int_0^\infty \left(\frac{1}{x}\mathbb{E}(A_\infty \wedge x) + \mathbb{P}(A_\infty > x)\right) dF(x) \\
&\leq \int_0^\infty \left(\frac{1}{x}\mathbb{E}(A_\infty 1_{A_\infty \leq x}) + 2\mathbb{P}(A_\infty > x)\right) dF(x) \\
&= 2\mathbb{E}[F(A_\infty)] + \mathbb{E}[A_\infty \int_{A_\infty}^\infty \frac{1}{x} dF(x)] = \mathbb{E}[\tilde{F}(A_\infty)]
\end{aligned}$$

where $\tilde{F}(x) = 2F(x) + x\int_x^\infty \frac{1}{u} dF(u)$. With $F(x) = x^k$ we have $\tilde{F}(x) = \frac{2-k}{1-k}x^k$ which finishes the proof. □

Notice that for $k \geq 1$, $\tilde{F} \equiv \infty$ so that the proposition above becomes useless. It can be shown that for $k = 1$ there is no universal constant c such that $\mathbb{E}[X_\infty^*] \leq c\mathbb{E}[A_\infty]$.

Now we procede to show how all these results imply theorem 2.1.

First let $X = (M^*)^2$ and $A = C_2\langle M, M\rangle$ where C_2 is a suitable constant such that $\mathbb{E}[X_T] \le \mathbb{E}[A_T]$ for any bounded stopping time (Such a constant exists thanks to Proposition 2.3). Then we deduce that for any $k \in (0, 1)$ we have

$$\mathbb{E}[(M^*)^{2k}_\infty] \le \frac{2-k}{1-k} C_2^k \mathbb{E}[\langle M, M\rangle^k_\infty]$$

So we just showed that for $p \in (0, 2)$ we have

$$\mathbb{E}[(M^*)^p_\infty] \le C_p \mathbb{E}[\langle M, M\rangle^{p/2}_\infty]$$

Now for the other inequality we consider the processes $X = \langle M, M\rangle^2$ and $A = C_4(M^*)^4$. Proposition 2.4 shows that A dominates X in the sense of Definition 2.5. Then again applying the last result we get for $k \in (0, 1)$

$$\mathbb{E}[\langle M, M\rangle^{2k}_\infty] \le \frac{2-k}{1-k} C_4^k \mathbb{E}[(M^*_\infty)^{4k}]$$

taking $p = 4k$ we have just seen that

$$c_p \mathbb{E}[\langle M, M\rangle^{p/2}_\infty] \le \mathbb{E}[(M^*_\infty)^p]$$

So the proof of theorem 2.1 is complete.

As a summary of the technique that is used: Itô's formula allowed us to show the two inequalities for most values of p and this domonation technique allowed us to extend the result to all values of $p > 0$. It should be mentioned that this is a very powerful principal in analysis that allows to show these types of statements and is very useful to have in your toolkit.

2.3. **Proof via time-change representation.** We notes that other proofs of BDG exist and in this section we give a sketch of a proof using the respresentation by time-change that we have seen in the last talk.

As we've seen before a continuous local martingale vanishing at 0 admits a time change under which it becomes a brownian motion. So that proving the BGD inequalities for Brownian motion will allow us to deduce the result for any continuous local martingale vanishing at 0. We shall not discuss this proof here, but instead refer to [RY13] for an alternative proof in the special case of Brownian motion.

3. **Conformal martingales and planar brownian motion**

In this section we study the two-dimensional local martingales in which the planar Brownian motion is a special case. For this purpose we use the complex notation. For instance the planar Brownian motion will be denoted $B = B^{(1)} + iB^{(2)}$ where $B^{(1)}, B^{(2)}$ are independent Brownian motions in one dimension. A complex local martingale in a process $Z = X + iY$ where X, Y are real local martingales.

Proposition 3.1. *If Z is a continuous complex martingale, there exists a unique continuous complex process of finite variation vanishing at 0 denoted by $\langle Z, Z\rangle$ such that $Z^2 - \langle Z, Z\rangle$ is a complex local martingale. Furthermore the following are equivalent*

(1) *Z^2 is a local martingale*
(2) $\langle Z, Z, \rangle = 0$
(3) $\langle X, X\rangle = \langle Y, Y\rangle$ *and* $\langle X, Y\rangle = 0$.

Proof. For the existence it suffices to define the braket by $\mathbb{C}$-linearity as

$$\langle Z, Z\rangle = \langle X, X\rangle - \langle Y, Y\rangle + 2i\langle X, Y\rangle$$

It is easy to check that this process satisfies all the desired conditions. For uniqueness we use the fact that a continuous martingale with fintie variation is constant. □

A local martingale satisfying the equivalent conditions above is called a conformal local martingale. The planar Brownian motion for instance is a conformal local martingale and if H is a complex valued locally bounded predictable process and Z a conformal local martingale then $U_t = \int_0^t H_s dZ_s$ is a conformal local martingale.

We recall the following differential operators from complex analysis

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and that a differentiable function $F : \mathbb{C} \to \mathbb{C}$ (in the sense of real coordiantes) is holomorphic if and only if $\frac{\partial F}{\partial \overline{z}} = 0$ in which case the $\mathbb{C}$-derivative is $F' = \frac{\partial F}{\partial z}$.

We have a similar looking result to Itô's formula in the complex case for conformal local martingales.

Proposition 3.2. *If Z is a conformal local martingale and F is a complex function on $\mathbb{C}$ which is twice differentiable (as a function of two real coordinates) then*

$$F(Z_t) = F(Z_0) + \int_0^t \frac{\partial F}{\partial z}(Z_s)dZ_s + \int_0^t \frac{\partial F}{\partial \overline{z}}(Z_s)d\overline{Z}_s + \frac{1}{4}\int_0^t \Delta F(Z_s)d\langle Z_s, \overline{Z}\rangle_s$$

References

[RY13] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.

U.C. Berkeley, Department of statistics, 335 Evans Hall Berkeley, CA 94720 USA
Email address: yassine.el-maazouz@berkeley.edu